The Rest-Frame Instant Form of Relativistic Perfect Fluids with Equation of State $\rho = \rho(n, s)$ and of Non-Dissipative Elastic Materials.

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Abstract

For perfect fluids with equation of state $\rho = \rho(n, s)$, Brown [1] gave an action principle depending only on their Lagrange coordinates $\alpha^i(x)$ without Clebsch potentials. After a reformulation on arbitrary spacelike hypersurfaces in Minkowski spacetime, the Wigner-covariant rest-frame instant form of these perfect fluids is given. Their Hamiltonian invariant mass can be given in closed form for the dust and the photon gas. The action for the coupling to tetrad gravity is given. Dixon's multipoles for the perfect fluids are studied on the rest-frame Wigner hyperplane. It is also shown that the same formalism can be applied to non-dissipative relativistic elastic materials described in terms of Lagrangian coordinates.

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I. INTRODUCTION

Stability of stellar models for rotating stars, gravity-fluid models, neutron stars, accretion discs around compact objects, collapse of stars, merging of compact objects are only some of the many topics in astrophysics and cosmology in which relativistic hydrodynamics is the basic underlying theory. This theory is also needed in heavy-ions collisions.

As shown in Ref. [1] there are many ways to describe relativistic perfect fluids by means of action functionals both in special and general relativity. Usually, besides the thermodynamical variables n (particle number density), ρ (energy density), p (pressure), T (temperature), s (entropy per particle), which are spacetime scalar fields whose values represent measurements made in the rest frame of the fluid (Eulerian observers), one characterizes the fluid motion by its unit timelike 4-velocity vector field U^μ (see Appendix A for a review of the relations among the local thermodynamical variables and Appendix B for a review of covariant relativistic thermodynamics following Ref. [2]). However, these variables are constrained due to [we use a general relativistic notation: "; μ" denotes a covariant derivative]
i) particle number conservation, $(nU^\mu)_{;\mu} = 0$;
ii) absence of entropy exchange between neighbouring flow lines, $(nsU^\mu)_{;\mu} = 0$;
iii) the requirement that the fluid flow lines should be fixed on the boundary.
Therefore one needs Lagrange multipliers to incorporate i) anf ii) into the action and this leads to use Clebsch (or velocity-potential) representations of the 4-velocity and action functionals depending on many redundant variables, generating first and second class constraints at the Hamiltonian level (see Appendix A).

Following Ref. [3] the previous constraint iii) may be enforced by replacing the unit 4-velocity U^μ with a set of spacetime scalar fields $\tilde{\alpha}^i(z)$, $i = 1, 2, 3$, interpreted as "Lagrangian (or comoving) coordinates for the fluid" labelling the fluid flow lines (physically determined by the average particle motions) passing through the points inside the boundary (on the boundary they are fixed: either the $\tilde{\alpha}^i(z^o, \vec{z})$'s have a compact boundary $V_\alpha(z^o)$ or they have assigned boundary conditions at spatial infinity). This requires the choice of an arbitrary spacelike hypersurface on which the α^i's are the 3-coordinates. A similar point of view is contained in the concept of "material space" of Refs. [4,5], describing the collection of all the idealized points of the material; besides to non-dissipative isentropic fluids the scheme can be applied to isotropic elastic media and anisotropic (crystalline) materials, namely to an arbitrary non-dissipative relativistic continuum [6]. See Ref. [7] for the study of the transformation from Eulerian to Lagrangian coordinates (in the non-relativistic framework of the Euler-Newton equations).

Notice that the use of Lagrangian (comoving) coordinates in place of Eulerian quantities allows the use of standard Poisson brackets in the Hamiltonian description, avoiding the formulation with Lie-Poisson brackets of Ref. [8], which could be recovered by a so-called Lagrangian to Eulerian map.

Let M^4 be a curved globally hyperbolic spacetime [with signature $\epsilon(+ - --)$, $\epsilon = \pm$] whose points have locally coordinates z^μ. Let ${}^4g_{\mu\nu}(z)$ be its 4-metric with determinant ${}^4g = |det\,{}^4g_{\mu\nu}|$. Given a perfect fluid with Lagrangian coordinates $\tilde{\alpha}(z) = \{\tilde{\alpha}^i(z)\}$, unit 4-velocity vector field $U^\mu(z)$ and particle number density $n(z)$, let us introduce the number flux vector

$$n(z)U^\mu(z) = \frac{J^\mu(\tilde{\alpha}^i(z))}{\sqrt{^4g(z)}}, \tag{1.1}$$

and the densitized fluid number flux vector or material current $[\epsilon^{0123} = 1/\sqrt{^4g};$ $\partial_\alpha(\sqrt{^4g}\epsilon^{\mu\nu\rho\sigma}) = 0;$ $\eta_{123}(\tilde{\alpha}^i)$ describes the orientation of the volume in thematerial space]

$$J^\mu(\tilde{\alpha}^i(z)) = -\sqrt{^4g}\epsilon^{\mu\nu\rho\sigma}\eta_{123}(\tilde{\alpha}^{\mathbf{i}}(z))\partial_\nu\tilde{\alpha}^1(z)\partial_\rho\tilde{\alpha}^2(z)\partial_\sigma\tilde{\alpha}^3(z),$$

$$\Rightarrow \quad n(z) = \frac{|J(\tilde{\alpha}^i(z))|}{\sqrt{^4g}} = \eta_{123}(\tilde{\alpha}^i(z))\frac{\sqrt{\epsilon^4 g_{\mu\nu}(z)J^\mu(\tilde{\alpha}^i(z))J^\nu(\tilde{\alpha}^i(z))}}{\sqrt{^4g(z)}},$$

$$\Rightarrow \quad \partial_\mu J^\mu(\tilde{\alpha}^i(z)) = \sqrt{^4g}\,[n(z)U^\mu(z)]_{;\mu} = 0,$$

$$\Rightarrow \quad J^\mu(\tilde{\alpha}^i(z))\partial_\mu\tilde{\alpha}^i(z) = [\sqrt{^4g}nU^\mu](z)\partial_\mu\tilde{\alpha}^i(z) = 0. \tag{1.2}$$

This shows that the fluid flow lines, whose tangent vector field is the fluid 4-velocity timelike vector field U^μ, are identified by $\tilde{\alpha}^{\mathbf{i}} = const.$ and that the particle number conservation is automatic. Moreover, if the entropy for particle is a function only of the fluid Lagrangian coordinates, $s = s(\tilde{\alpha}^{\mathbf{i}})$, the assumed form of J^μ also implies automatically the absence of entropy exchange between neighbouring flow lines, $(nsU^\mu)_{,\mu} = 0$. Since $U^\mu\partial_\mu s(\tilde{\alpha}^i) = 0$, the perfect fluid is locally adiabatic; instead for an isentropic fluid we have $\partial_\mu s = 0$, namely $s = const..$

Even if in general the timelike vector field $U^\mu(z)$ is not surface forming (namely has a non-vanishing vorticity, see for instance Ref. [9]), in each point z we can consider the spacelike hypersurface orthogonal to the fluid flow line in that point (namely we split the tangent space $T_z M^4$ at z in the $U^\mu(z)$ direction and in the orthogonal complement) and consider $\frac{1}{3!}[U^\mu\epsilon_{\mu\nu\rho\sigma}dz^\nu \wedge dz^\rho \wedge dz^\sigma](z)$ as the infinitesimal 3-volume on it at z. Then the 3-form

$$\eta[z] = [\eta_{123}(\tilde{\alpha})d\tilde{\alpha}^1 \wedge d\tilde{\alpha}^2 \wedge d\tilde{\alpha}^3](z) = \frac{1}{3!}n(z)[U^\mu\epsilon_{\mu\nu\rho\sigma}dz^\nu \wedge dz^\rho \wedge dz^\sigma](z), \tag{1.3}$$

may be interpreted as the number of particles in this 3-volume. If V is a volume around z on the spacelike hypersurface, then $\int_V \eta$ is the number of particle in V and $\int_V s\eta$ is the total entropy contained in the flow lines included in the volume V. Note that locally η_{123} can be set to unity by an appropriate choice of coordinates.

In Ref. [1] it is shown that the action functional

$$S[^4g_{\mu\nu}, \tilde{\alpha}] = -\int d^4z\sqrt{^4g(z)}\,\rho(\frac{|J(\tilde{\alpha}^i(z))|}{\sqrt{^4g(z)}}, s(\tilde{\alpha}^{\mathbf{i}}(z))), \tag{1.4}$$

has a variation with respect to the 4-metric, which gives rise to the correct stress tensor $T^{\mu\nu} = (\rho + p)U^\mu U^\nu - \epsilon\,p\,^4g^{\mu\nu}$ with $p = n\frac{\partial\rho}{\partial n}|_s - \rho$ for a perfect fluid [see Appendix A].

The Euler-Lagrange equations associated to the variation of the Lagrangian coordinates are [1] [$V_\mu = \mu U_\mu$ is the Taub current, see Appendix A]

$$\frac{1}{\sqrt{^4g}}\frac{\delta S}{\delta\tilde{\alpha}^i} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}V_{\mu;\nu}\,\eta_{ijk}\,\partial_\rho\tilde{\alpha}^j\,\partial_\sigma\tilde{\alpha}^k - nT\frac{\partial s}{\partial\tilde{\alpha}^i} = 0,$$

$$\frac{1}{\sqrt{^4g}}\frac{\delta S}{\delta\tilde{\alpha}^i}\partial_\mu\tilde{\alpha}^i = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}V_{\alpha;\beta}\,U^\nu\,\epsilon_{\nu\mu\gamma\delta} - T\,\partial_\mu s = 2V_{[\mu;\nu]}U^\nu - T\,\partial_\mu s = 0. \tag{1.5}$$

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As shown in Appendix A, these equations together with the entropy exchange constraint imply the Euler equations implied from the conservation of the stress-energy-momentum tensor.

Therefore, with this description the conservation laws are automatically satisfied and the Euler-Lagrange equations are equivalent to the Euler equations. In Minkowski spacetime the conserved particle number is $\mathcal{N} = \int_{V_\alpha(z^o)} d^3z\, n(z)\, U^o(z) = \int_{V_\alpha(z^o)} d^3z\, J^o(\tilde{\alpha}^i(z))$, while the conserved entropy per particle is $\int_{V_\alpha(z^o)} d^3z\, s(z)\, n(z)\, U^o(z) = \int_{V_\alpha(z^o)} d^3z\, s(z)\, J^o(\tilde{\alpha}^i)$. Moreover, the conservation laws $T^{\mu\nu}{}_{,\nu} = 0$ will generate the conserved 4-momentum and angular momentum of the fluid.

However, in Ref. [1] there are only some comments on the Hamiltonian description implied by this particular action.

This description of perfect fluids fits naturally with parametrized Minkowski theories [10] for arbitrary isolated relativistic systems [see Ref. [11] for a review] on arbitrary spacelike hypersurfaces, leaves of the foliation of Minkowski spacetime M^4 associated with one of its 3+1 splittings.

Therefore, the aim of this paper is to find the Wigner covariant rest-frame instant form of the dynamics of a perfect fluid, on the special Wigner hyperplanes orthogonal to the total 4-momentum of the fluid. In this way we will get the description of the global rest frame of the fluid as a whole; instead, the 4-velocity vector field U^μ defines the local rest frame in each point of the fluid by means of the projector ${}^4g^{\mu\nu} - \epsilon U^\mu U^\nu$. This approach will also produce automatically the coupling of the fluid to ADM metric and tetrad gravity with the extra property of allowing a well defined deparametrization of the theory leading to the rest-frame instant form in Minkowski spacetime with Cartesian coordinates when we put equal to zero the Newton constant G [11]. In this paper we will consider the perfect fluid only in Minkowski spacetime, except for some comments on its coupling to gravity.

The starting point is the foliation of Minkowski spacetime M^4, which is defined by an embedding $R \times \Sigma \to M^4$, $(\tau, \vec{\sigma}) \mapsto z^\mu(\tau, \vec{\sigma}) \in \Sigma_\tau$ and with Σ an abstract 3-surface diffeomorphic to R^3, with Σ_τ its copy embedded in M^4 labelled by the value τ (the scalar mathematical "time" parameter τ labels the leaves of the foliation, $\vec{\sigma}$ are curvilinear coordinates on Σ_τ and $\sigma^{\check{A}} = (\sigma^\tau = \tau, \sigma^{\check{r}})$ are Σ_τ-adapted holonomic coordinates for M^4). See Appendix C for the notations on spacelike hypersurfaces.

In this way one gets a parametrized field theory with a covariant 3+1 splitting of Minkowski spacetime and already in a form suited to the transition to general relativity in its ADM canonical formulation (see also Ref. [12], where a theoretical study of this problem is done in curved spacetimes). The price is that one has to add as new independent configuration variables the embedding coordinates $z^\mu(\tau, \vec{\sigma})$ of the points of the spacelike hypersurface Σ_τ [the only ones carrying Lorentz indices] and then to define the fields on Σ_τ so that they know the hypersurface Σ_τ of τ-simultaneity [for a Klein-Gordon field $\phi(x)$, this new field is $\tilde{\phi}(\tau, \vec{\sigma}) = \phi(z(\tau, \vec{\sigma}))$: it contains the non-local information about the embedding]. Then one rewrites the Lagrangian of the given isolated system in the form required by the coupling to an external gravitational field, makes the previous 3+1 splitting of Minkowski spacetime and interprets all the fields of the system as the new fields on Σ_τ (they are Lorentz scalars, having only surface indices). Instead of considering the 4-metric as describing a gravitational field (and therefore as an independent field as it is done in metric gravity, where one adds the Hilbert action to the action for the matter fields), here one

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replaces the 4-metric with the the induced metric $g_{\check{A}\check{B}}[z] = z_{\check{A}}^\mu \eta_{\mu\nu} z_{\check{B}}^\nu$ on Σ_τ [a functional of z^μ; $z_{\check{A}}^\mu = \partial z^\mu / \partial \sigma^{\check{A}}$ are flat inverse tetrad fields on Minkowski spacetime with the $z_{\check{r}}^\mu$'s tangent to Σ_τ] and considers the embedding coordinates $z^\mu(\tau, \vec\sigma)$ as independent fields [this is not possible in metric gravity, because in curved spacetimes $z_{\check{A}}^\mu \neq \partial z^\mu / \partial \sigma^{\check{A}}$ are not tetrad fields so that holonomic coordinates $z^\mu(\tau, \vec\sigma)$ do not exist]. From this Lagrangian, besides a Lorentz-scalar form of the constraints of the given system, we get four extra primary first class constraints

$$\mathcal{H}_\mu(\tau, \vec\sigma) = \rho_\mu(\tau, \vec\sigma) - l_\mu(\tau, \vec\sigma) T_{sys}^{\tau\tau}(\tau, \vec\sigma) - z_{\check{r}\mu}(\tau, \vec\sigma) T_{sys}^{\check{r}\tau}(\tau, \vec\sigma) \approx 0, \qquad (1.6)$$

[here $T_{sys}^{\tau\tau}(\tau, \vec\sigma) = \mathcal{M}(\tau, \vec\sigma)$, $T_{sys}^{\check{r}\tau}(\tau, \vec\sigma) = \mathcal{M}^{\check{r}}(\tau, \vec\sigma)$, are the components of the energy-momentum tensor in the holonomic coordinate system, corresponding to the energy- and momentum-density of the isolated system; one has $\{\mathcal{H}_{(\mu)}(\tau, \vec\sigma), \mathcal{H}_{(\nu)}(\tau, \vec\sigma')\} = 0$] implying the independence of the description from the choice of the 3+1 splitting, i.e. from the choice of the foliation with spacelike hypersufaces. As shown in Appendix C the evolution vector is given by $z_\tau^\mu = N_{[z](flat)} l^\mu + N_{[z](flat)}^{\check{r}} z_{\check{r}}^\mu$, where $l^\mu(\tau, \vec\sigma)$ is the normal to Σ_τ in $z^\mu(\tau, \vec\sigma)$ and $N_{[z](flat)}(\tau, \vec\sigma)$, $N_{[z](flat)}^{\check{r}}(\tau, \vec\sigma)$ are the flat lapse and shift functions defined through the metric like in general relativity: however, now they are not independent variables but functionals of $z^\mu(\tau, \vec\sigma)$.

The Dirac Hamiltonian contains the piece $\int d^3\sigma \lambda^\mu(\tau, \vec\sigma) \mathcal{H}_\mu(\tau, \vec\sigma)$ with $\lambda^\mu(\tau, \vec\sigma)$ Dirac multipliers. It is possible to rewrite the integrand in the form [$\gamma^{\check{r}\check{s}} = -\epsilon\,^3 g^{\check{r}\check{s}}$ is the inverse of the spatial metric $g_{\check{r}\check{s}} = \,^4 g_{\check{r}\check{s}} = -\epsilon\,^3 g_{\check{r}\check{s}}$, with $^3 g_{\check{r}\check{s}}$ of positive signature $(+++)$]

$$\lambda_\mu(\tau, \vec\sigma) \mathcal{H}^\mu(\tau, \vec\sigma) = [(\lambda_\mu l^\mu)(l_\nu \mathcal{H}^\nu) - (\lambda_\mu z_r^\mu)(\gamma^{\check{r}\check{s}} z_{\check{s}\nu} \mathcal{H}^\nu)](\tau, \vec\sigma) \stackrel{def}{=}$$
$$\stackrel{def}{=} N_{(flat)}(\tau, \vec\sigma)(l_\mu \mathcal{H}^\mu)(\tau, \vec\sigma) - N_{(flat)\check{r}}(\tau, \vec\sigma)(\gamma^{\check{r}\check{s}} z_{\check{s}\nu} \mathcal{H}^\nu)(\tau, \vec\sigma), \qquad (1.7)$$

with the (non-holonomic form of the) constraints $(l_\mu \mathcal{H}^\mu)(\tau, \vec\sigma) \approx 0$, $(\gamma^{\check{r}\check{s}} z_{\check{s}\mu} \mathcal{H}^\mu)(\tau, \vec\sigma) \approx 0$, satisfying the universal Dirac algebra of the ADM constraints. In this way we have defined new flat lapse and shift functions

$$N_{(flat)}(\tau, \vec\sigma) = \lambda_\mu(\tau, \vec\sigma) l^\mu(\tau, \vec\sigma),$$
$$N_{(flat)\check{r}}(\tau, \vec\sigma) = \lambda_\mu(\tau, \vec\sigma) z_{\check{r}}^\mu(\tau, \vec\sigma). \qquad (1.8)$$

which have the same content of the arbitrary Dirac multipliers $\lambda_\mu(\tau, \vec\sigma)$, namely they multiply primary first class constraints satisfying the Dirac algebra. In Minkowski spacetime they are quite distinct from the previous lapse and shift functions $N_{[z](flat)}$, $N_{[z](flat)\check{r}}$, defined starting from the metric. Since the Hamilton equations imply $z_\tau^\mu(\tau, \vec\sigma) = \lambda^\mu(\tau, \vec\sigma)$, it is only through the equations of motion that the two types of functions are identified. Instead in general relativity the lapse and shift functions defined starting from the 4-metric are the coefficients (in the canonical part H_c of the Hamiltonian) of secondary first class constraints satisfying the Dirac algebra independently from the equations of motion.

For the relativistic perfect fluid with equation of state $\rho = \rho(n, s)$ in Minkowski spacetime, we have only to replace the external 4-metric $^4 g_{\mu\nu}$ with $g_{\check{A}\check{B}}(\tau, \vec\sigma) = \,^4 g_{\check{A}\check{B}}(\tau, \vec\sigma)$ and the scalar fields for the Lagrangian coordinates with $\alpha^i(\tau, \vec\sigma) = \tilde\alpha^i(z(\tau, \vec\sigma))$; now either the $\alpha^i(\tau, \vec\sigma)$'s have a compact boundary $V_\alpha(\tau) \subset \Sigma_\tau$ or have boundary conditions at spatial infinity. For each value of τ, one could invert $\alpha^i = \alpha^i(\tau, \vec\sigma)$ to $\vec\sigma = \vec\sigma(\tau, \alpha^i)$ and use the α^i's as a special coordinate system on Σ_τ inside the support $V_\alpha(\tau) \subset \Sigma_\tau$: $z^\mu(\tau, \vec\sigma(\tau, \alpha^i)) = \check{z}^\mu(\tau, \alpha^i)$.

By going to Σ_τ-adapted coordinates such that $\eta_{123}(\alpha) = 1$ we get $[\gamma = |det\, g_{\check{r}\check{s}}|; \sqrt{g} = \sqrt{{}^4 g} = \sqrt{|det\, g_{\check{A}\check{B}}|} = N\sqrt{\gamma}]$

$$J^{\check{A}}(\alpha^i(\tau, \vec{\sigma})) = [N\sqrt{\gamma}\, n\, U^{\check{A}}](\tau, \vec{\sigma}),$$

$$J^\tau(\alpha^i(\tau, \vec{\sigma})) = [-\epsilon^{\check{r}\check{u}\check{v}} \partial_{\check{r}}\alpha^1 \partial_{\check{u}}\alpha^2 \partial_{\check{v}}\alpha^3](\tau, \vec{\sigma}) = -det\,(\partial_{\check{r}}\alpha^i)(\tau, \vec{\sigma}),$$

$$J^{\check{r}}(\alpha^i(\tau, \vec{\sigma})) = [\sum_{i=1; i,j,k\, cyclic}^{3} \partial_\tau \alpha^i \epsilon^{\check{r}\check{u}\check{v}} \partial_{\check{u}}\alpha^j \partial_{\check{v}}\alpha^k](\tau, \vec{\sigma}) =$$

$$= \frac{1}{2}\epsilon^{\check{r}\check{u}\check{v}}\epsilon_{ijk}[\partial_\tau \alpha^i \partial_{\check{u}}\alpha^j \partial_{\check{v}}\alpha^k](\tau, \vec{\sigma}),$$

$$\Rightarrow \quad n(\tau, \vec{\sigma}) = \frac{|J|}{N\sqrt{\gamma}}(\tau, \vec{\sigma}) = \frac{\sqrt{\epsilon g_{\check{A}\check{B}} J^{\check{A}} J^{\check{B}}}}{N\sqrt{\gamma}}(\tau, \vec{\sigma}), \quad (1.9)$$

with $\mathcal{N} = \int_{V_\alpha(\tau)} d^3\sigma J^\tau(\alpha^i(\tau, \vec{\sigma}))$ giving the conserved particle number and $\int_{V_\alpha(\tau)} d^3\sigma(s\, J^\tau)(\tau, \vec{\sigma})$ giving the conserved entropy per particle.

The action becomes

$$S = \int d\tau d^3\sigma L(z^\mu(\tau, \vec{\sigma}), \alpha^i(\tau, \vec{\sigma})) =$$

$$= -\int d\tau d^3\sigma (N\sqrt{\gamma})(\tau, \vec{\sigma}) \rho(\frac{|J(\alpha^i(\tau, \vec{\sigma}))|}{(N\sqrt{\gamma})(\tau\vec{\sigma})}, s(\alpha^i(\tau, \vec{\sigma}))) =$$

$$= -\int d\tau d^3\sigma (N\sqrt{\gamma})(\tau, \vec{\sigma})$$

$$\rho(\frac{1}{\sqrt{\gamma(\tau, \vec{\sigma})}}\sqrt{[(J^\tau)^2 - {}^3 g_{\check{u}\check{v}}\frac{J^{\check{u}} + N^{\check{u}} J^\tau}{N}\frac{J^{\check{v}} + N^{\check{v}} J^\tau}{N}]}(\tau, \vec{\sigma}; \alpha^i(\tau, \vec{\sigma})),\, s(\alpha^i(\tau, \vec{\sigma}))),$$

$$(1.10)$$

with $N = N_{[z](flat)}$, $N^{\check{r}} = N^{\check{r}}_{[z](flat)}$.

This is the form of the action whose Hamiltonian formulation will be studied in this paper.

We shall begin in Section II with the simple case of dust, whose equation of state is $\rho = \mu n$.

In Section III we will define the "external" and "internal" centers of mass of the dust.

In Section IV we will study Dixon's multipoles of a perfect fluid on the Wigner hyperplane in Minkowski spacetime using the dust as an example.

Then in Section V we will consider some equations of state for isentropic fluids and we will make some comments on non-isentropic fluids.

In Section VI we will define the coupling to ADM metric and tetrad gravity.

In Section VII we will describe with the same technology isentropic elastic media.

In the Conclusions, after some general remarks, we will delineate the treatment of perfect fluids in tetrad gravity (this will be the subject of a future paper).

In Appendix A there is a review of some of the results of Ref. [1] for relativistic perfect fluids.

In Appendix B there is a review of covariant relativistic thermodynamics of equilibrium and non-equilibrium.

In Appendix C there is some notation on spacelike hypersurfaces.

In Appendix D there is the definition of other types of Dixon's multipoles.

II. DUST.

Let us consider first the simplest case of an isentropic perfect fluid, a dust with $p = 0$, $s = const.$, and equation of state $\rho = \mu n$. In this case the chemical potential μ is the rest mass-energy of a fluid particle: $\mu = m$ (see Appendix A).

Eq.(1.10) implies that the Lagrangian density is [we shall use the notation $g_{\check{A}\check{B}} = {}^4g_{\check{A}\check{B}}$ with signature $\epsilon(+---)$, $\epsilon = \pm 1$; by using the notation with lapse and shift functions given in Appendix C we get: $g_{\tau\tau} = \epsilon(N^2 - {}^3g_{\check{r}\check{s}}N^{\check{r}}N^{\check{s}})$, $g_{\tau\check{r}} = -\epsilon\,{}^3g_{\check{r}\check{s}}N^{\check{s}}$, $g_{\check{r}\check{s}} = -\epsilon\,{}^3g_{\check{r}\check{s}}$ with ${}^3g_{\check{r}\check{s}}$ of positive signature $(+++)$, $g^{\tau\tau} = \frac{\epsilon}{N^2}$, $g^{\tau\check{r}} = -\epsilon\frac{N^{\check{r}}}{N^2}$, $g^{\check{r}\check{s}} = -\epsilon({}^3g^{\check{r}\check{s}} - \frac{N^{\check{r}}N^{\check{s}}}{N^2})$; the inverse of the spatial 4-metric ${}^4g_{\check{r}\check{s}}$ is denoted $\gamma^{\check{r}\check{s}} = {}^4\gamma^{\check{r}\check{s}} = -\epsilon\,{}^3g^{\check{r}\check{s}}$, where ${}^3g^{\check{r}\check{s}}$ is the inverse of the 3-metric ${}^3g_{\check{r}\check{s}}$ and we use $\sqrt{\gamma} = \sqrt{{}^3g_{\check{r}\check{s}}}$]

$$L(\alpha^i, z^\mu) = -\sqrt{g}\rho = -\mu\sqrt{g}n = -\mu\sqrt{\epsilon g_{\check{A}\check{B}}J^{\check{A}}J^{\check{B}}} =$$
$$= -\mu N\sqrt{(J^\tau)^2 - {}^3g_{\check{r}\check{s}}Y^{\check{r}}Y^{\check{s}}} = -\mu NX,$$

$$Y^{\check{r}} = \frac{1}{N}(J^{\check{r}} + N^{\check{r}}J^\tau),$$
$$X = \sqrt{(J^\tau)^2 - {}^3g_{\check{r}\check{s}}Y^{\check{r}}Y^{\check{s}}} = \frac{\sqrt{g}}{N}n = \sqrt{\gamma}n, \tag{2.1}$$

with J^τ, $J^{\check{r}}$ given in Eqs.(1.9).

The momentum conjugate to α^i is

$$\Pi_i = \frac{\partial L}{\partial\partial_\tau\alpha^i} = \mu\frac{Y^{\check{t}}\,{}^3g_{\check{t}\check{r}}\,\epsilon^{\check{r}\check{u}\check{v}}\,\partial_{\check{u}}\alpha^j\partial_{\check{v}}\alpha^k}{X}\Big|_{i,j,k\quad cyclic} =$$
$$= \mu\frac{Y^{\check{t}}}{2X}\,{}^3g_{\check{t}\check{r}}\,\epsilon^{\check{r}\check{u}\check{v}}\epsilon_{ijk}\,\partial_{\check{u}}\alpha^j\partial_{\check{v}}\alpha^k = \mu\frac{Y^{\check{r}}}{X}\,T_{\check{r}i},$$

$$T_{\check{t}i} \stackrel{def}{=} \frac{1}{2}g_{\check{t}\check{r}}\,\epsilon^{\check{r}\check{u}\check{v}}\epsilon_{ijk}\,\partial_{\check{u}}\alpha^j\partial_{\check{v}}\alpha^k = g_{\check{t}\check{r}}\,(ad\,J_{i\check{r}}), \tag{2.2}$$

where $ad\,J_{i\check{r}} = (det\,J)J_{i\check{r}}^{-1}$ is the adjoint matrix of the Jacobian $J = (J_{i\check{r}} = \partial_{\check{r}}\alpha^i)$ of the transformation from the Lagrangian coordinates $\alpha^i(\tau, \vec{\sigma})$ to the Eulerian ones $\vec{\sigma}$ on Σ_τ.

The momentum conjugate to z^μ is

$$\rho_\mu(\tau, \vec{\sigma}) = -\frac{\partial L}{\partial z_\tau^\mu}(\tau, \vec{\sigma}) = \Big[\mu\,l_\mu\frac{(J^\tau)^2}{X} + \mu\,z_{r\mu}J^\tau\frac{Y^r}{X}\Big](\tau, \vec{\sigma}). \tag{2.3}$$

The following Poisson brackets are assumed

$$\{z^\mu(\tau, \vec{\sigma}), \rho_\nu(\tau, \vec{\sigma}')\} = -\eta_\nu^\mu\delta^3(\vec{\sigma} - \vec{\sigma}'),$$
$$\{\alpha^i(\tau, \vec{\sigma}), \Pi_j(\tau, \vec{\sigma}')\} = \delta_j^i\delta^3(\vec{\sigma} - \vec{\sigma}'). \tag{2.4}$$

We can express $Y^{\check{r}}/X$ in terms of Π_i with the help of the inverse $(T^{-1})^{\check{r}i}$ of the matrix $T_{\check{t}i}$

$$\frac{Y^{\check{r}}}{X} = \frac{1}{\mu}(T^{-1})^{\check{r}i}\Pi_i, \tag{2.5}$$

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where

$$(T^{-1})^{\check{r}i} = \frac{{}^3g^{\check{r}\check{s}}\, \partial_{\check{s}}\alpha^i}{det\,(\partial_{\check{u}}\alpha^k)}. \tag{2.6}$$

From the definition of X we find

$$X = \frac{\mu J^\tau}{\sqrt{\mu^2 + {}^3g_{\check{u}\check{v}}(T^{-1})^{\check{u}i}(T^{-1})^{\check{v}j}\Pi_i\Pi_j}}. \tag{2.7}$$

Consequently, we can get the expression of the velocities of the Lagrangian coordinates in terms of the momenta

$$\partial_\tau\alpha^i = -\frac{J^{\check{r}}\partial_{\check{r}}\alpha^i}{J^\tau} = \frac{(N^{\check{r}}J^\tau - NY^{\check{r}})\,\partial_{\check{r}}\alpha^i}{J^\tau}, \tag{2.8}$$

namely

$$\partial_\tau\alpha^i = \partial_{\check{r}}\alpha^i \left[N^{\check{r}} - N(T^{-1})^{\check{r}i}\Pi_i \sqrt{\mu^2 + {}^3g_{\check{u}\check{v}}(T^{-1})^{\check{u}i}(T^{-1})^{\check{v}j}\Pi_i\Pi_j} \right]. \tag{2.9}$$

Now ρ_μ can be expressed as a function of the z's, α's and Π's:

$$\rho_\mu = l_\mu\, J^\tau \sqrt{\mu^2 + {}^3g_{\check{u}\check{v}}(T^{-1})^{\check{u}i}\Pi_i(T^{-1})^{\check{v}j}\Pi_j} + z_{\check{r}\mu}\, J^\tau\, (T^{-1})^{\check{r}i}\Pi_i. \tag{2.10}$$

Since the Lagrangian is homogenous in the velocities, the Hamiltonian is only

$$H_D = \int d^3\sigma\lambda^\mu(\tau,\vec{\sigma})\mathcal{H}_\mu(\tau,\vec{\sigma}), \tag{2.11}$$

where the $\mathcal{H}_\mu$ are the primary constraints

$$\mathcal{H}_\mu = \rho_\mu - l_\mu\,\mathcal{M} + z_{\check{r}\mu}\,\mathcal{M}^{\check{R}} \approx 0,$$

$$\mathcal{M} = T^{\tau\tau} = J^\tau \sqrt{\mu^2 + {}^3g_{\check{u}\check{v}}(T^{-1})^{\check{u}i}\Pi_i(T^{-1})^{\check{v}j}\Pi_j},$$
$$\mathcal{M}^{\check{r}} = T^{\tau\check{r}} = J^\tau\,(T^{-1})^{\check{r}i}\Pi_i. \tag{2.12}$$

satisfying

$$\{\mathcal{H}_\mu(\tau,\vec{\sigma}), \mathcal{H}_\nu(\tau,\vec{\sigma}^{'})\} = 0. \tag{2.13}$$

One finds that $\{\mathcal{H}_\mu(\tau,\vec{\sigma}), H_D\} = 0$. Therefore, there are only the four first class constraints $\mathcal{H}_\mu(\tau,\vec{\sigma}) \approx 0$. They describe the arbitrariness of the foliation: physical results do not depend on its choice.

The conserved Poincaré generators are (the suffix "s" denotes the hypersurface Σ_τ)

$$p_s^\mu = \int d^3\sigma\rho^\mu(\tau,\vec{\sigma}),$$
$$J_s^{\mu\nu} = \int d^3\sigma[z^\mu(\tau,\vec{\sigma})\rho^\nu(\tau,\vec{\sigma}) - z^\nu(\tau,\vec{\sigma})\rho^\mu(\tau,\vec{\sigma})], \tag{2.14}$$

and one has

$$\{z^\mu(\tau, \vec{\sigma}), p_s^\nu\} = -\eta^{\mu\nu}, \tag{2.15}$$

$$\int d^3\sigma \mathcal{H}_\mu(\tau, \vec{\sigma}) = p_s^\mu - \int d^3\sigma \Big[l_\mu\, J^\tau\, \sqrt{\mu^2 + {}^3g_{\check{u}\check{v}}(T^{-1})^{\check{u}i}\Pi_i (T^{-1})^{\check{v}j}\Pi_j} \Big](\tau, \vec{\sigma}) +$$

$$+ \int d^3\sigma \Big[z_{\check{r}\mu}\, J^\tau\, (T^{-1})^{\check{r}i}\Pi_i \Big](\tau, \vec{\sigma}) \approx 0. \tag{2.16}$$

Let us now restrict ourselves to spacelike hyperplanes Σ_τ by imposing the gauge-fixings

$$\zeta^\mu(\tau, \vec{\sigma}) = z^\mu(\tau, \vec{\sigma}) - x_s^\mu(\tau) - b_{\check{r}}^\mu(\tau)\sigma^{\check{r}} \approx 0,$$

$$\{\zeta^\mu(\tau, \vec{\sigma}), \mathcal{H}_\nu(\tau, \vec{\sigma}')\} = -\eta_\nu^\mu \delta^3(\vec{\sigma} - \vec{\sigma}'), \tag{2.17}$$

where $x_s^\mu(\tau)$ is an arbitrary point of Σ_τ, chosen as origin of the coordinates $\sigma^{\check{r}}$, and $b_{\check{r}}^\mu(\tau)$, $\check{r} = 1, 2, 3$, are three orthonormal vectors such that the constant (future pointing) normal to the hyperplane is

$$l^\mu(\tau, \vec{\sigma}) \approx l^\mu = b_\tau^\mu = \epsilon^\mu{}_{\alpha\beta\gamma} b_{\check{1}}^\alpha(\tau) b_{\check{2}}^\beta(\tau) b_{\check{3}}^\gamma(\tau). \tag{2.18}$$

Therefore, we get

$$z_{\check{r}}^\mu(\tau, \vec{\sigma}) \approx b_{\check{r}}^\mu(\tau),$$
$$z_\tau^\mu(\tau, \vec{\sigma}) \approx \dot{x}_s^\mu(\tau) + \dot{b}_{\check{r}}^\mu(\tau)\sigma^{\check{r}},$$
$$g_{\check{r}\check{s}}(\tau, \vec{\sigma}) \approx -\epsilon\delta_{\check{r}\check{s}}, \qquad \gamma^{\check{r}\check{s}}(\tau, \vec{\sigma}) \approx -\epsilon\delta^{\check{r}\check{s}}, \qquad \gamma(\tau, \vec{\sigma}) \approx 1. \tag{2.19}$$

By introducing the Dirac brackets for the resulting second class constraints

$$\{A, B\}^* = \{A, B\} - \int d^3\sigma [\{A, \zeta^\mu(\tau, \vec{\sigma})\}\{\mathcal{H}_\mu(\tau, \vec{\sigma}), B\} - \{A, \mathcal{H}_\mu(\tau, \vec{\sigma})\}\{\zeta^\mu(\tau, \vec{\sigma}), B\}], \tag{2.20}$$

we find that, by using Eq.(2.15) and (2.16) [with $x_s^\mu(\tau) = z^\mu(\tau, \vec{\sigma}) - b_{\check{r}}^\mu(\tau)\sigma^{\check{r}} - \zeta^\mu(\tau, \vec{\sigma})$ and with the assumption $\{b_{\check{r}}^\mu(\tau), p_s^\nu\} = 0$], we get

$$\{x_s^\mu(\tau), p_s^\nu(\tau)\}^* = -\eta^{\mu\nu}. \tag{2.21}$$

The ten degrees of freedom describing the hyperplane are $x_s^\mu(\tau)$ with conjugate momentum p_s^μ and six variables $\phi_\lambda(\tau)$, $\lambda = 1, .., 6$, which parametrize the orthonormal tetrad $b_{\check{A}}^\mu(\tau)$, with their conjugate momenta $T_\lambda(\tau)$.

The preservation of the gauge-fixings $\zeta^\mu(\tau, \vec{\sigma}) \approx 0$ in time implies

$$\frac{d}{d\tau}\zeta^\mu(\tau, \vec{\sigma}) = \{\zeta^\mu(\tau, \vec{\sigma}), H_D\} = -\lambda^\mu(\tau, \vec{\sigma}) - \dot{x}_s^\mu(\tau) - \dot{b}_{\check{r}}^\mu(\tau)\sigma^{\check{r}} \approx 0, \tag{2.22}$$

so that one has [by using $\dot{b}_\tau^\mu = 0$ and $\dot{b}_{\check{r}}^\mu(\tau)b_{\check{r}}^\nu(\tau) = -b_{\check{r}}^\mu(\tau)\dot{b}_{\check{r}}^\nu(\tau)$]

$$\lambda^\mu(\tau, \vec{\sigma}) \approx \tilde{\lambda}^\mu(\tau) + \tilde{\lambda}^\mu{}_\nu(\tau)b_{\check{r}}^\nu(\tau)\sigma^{\check{r}},$$
$$\tilde{\lambda}^\mu(\tau) = -\dot{x}_s^\mu(\tau),$$
$$\tilde{\lambda}^{\mu\nu}(\tau) = -\tilde{\lambda}^{\nu\mu}(\tau) = \frac{1}{2}[\dot{b}_{\check{r}}^\mu(\tau)b_{\check{r}}^\nu(\tau) - b_{\check{r}}^\mu(\tau)\dot{b}_{\check{r}}^\nu(\tau)]. \tag{2.23}$$

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Thus, the Dirac Hamiltonian becomes

$$H_D = \tilde{\lambda}^\mu(\tau)\tilde{\mathcal{H}}_\mu(\tau) - \frac{1}{2}\tilde{\lambda}^{\mu\nu}(\tau)\tilde{\mathcal{H}}_{\mu\nu}(\tau), \tag{2.24}$$

and this shows that the gauge fixings $\zeta^\mu(\tau,\vec{\sigma}) \approx 0$ do not transform completely the constraints $\mathcal{H}_\mu(\tau,\vec{\sigma}) \approx 0$ in their second class partners; still the following ten first class constraints are left

$$\tilde{\mathcal{H}}^\mu(\tau) = \int d^3\sigma \mathcal{H}^\mu(\tau,\vec{\sigma}) = p_s^\mu - $$
$$- \int d^3\sigma \Big(J^\tau \Big[l^\mu \sqrt{\mu^2 + \delta_{\check{u}\check{v}}(T^{-1})^{\check{u}i}\Pi_i(T^{-1})^{\check{v}j}\Pi_j} + b_{\check{r}}^\mu (T^{-1})^{\check{r}l}\Pi_l \Big] \Big)(\tau,\vec{\sigma}) \approx 0,$$

$$\tilde{\mathcal{H}}^{\mu\nu}(\tau) = b_{\check{r}}^\mu(\tau) \int d^3\sigma \sigma^{\check{r}} \mathcal{H}^\nu(\tau,\vec{\sigma}) - b_{\check{r}}^\nu(\tau) \int d^3\sigma \sigma^{\check{r}} \mathcal{H}^\mu(\tau,\vec{\sigma}) = $$
$$= S_s^{\mu\nu}(\tau) - $$
$$- [b_{\check{r}}^\mu(\tau)b_\tau^\nu - b_{\check{r}}^\nu(\tau)b_\tau^\mu] \int d^3\sigma \sigma^{\check{r}} \Big(J^\tau \sqrt{\mu^2 + \delta_{\check{u}\check{v}}(T^{-1})^{\check{u}i}\Pi_i(T^{-1})^{\check{v}j}\Pi_j} \Big)(\tau,\vec{\sigma}) + $$
$$+ [b_{\check{r}}^\mu(\tau)b_{\check{s}}^\nu(\tau) - b_{\check{r}}^\nu(\tau)b_{\check{s}}^\mu(\tau)] \int d^3\sigma \sigma^{\check{r}} \Big(J^\tau (T^{-1})^{\check{s}l}\Pi_l \Big)(\tau,\vec{\sigma}) \approx 0. \tag{2.25}$$

Here $S_s^{\mu\nu}$ is the spin part of the Lorentz generators

$$J_s^{\mu\nu} = x_s^\mu p_s^\nu - x_s^\nu p_s^\mu + S_s^{\mu\nu},$$
$$S_s^{\mu\nu} = b_{\check{r}}^\mu(\tau) \int d^3\sigma \sigma^{\check{r}}\rho^\nu(\tau,\vec{\sigma}) - b_{\check{r}}^\nu(\tau) \int d^3\sigma \sigma^{\check{r}}\rho^\mu(\tau,\vec{\sigma}). \tag{2.26}$$

As shown in Ref. [10] instead of finding $\phi_\lambda(\tau), T_\lambda(\tau)$, one can use the redundant variables $b_{\check{A}}^\mu(\tau), S_s^{\mu\nu}(\tau)$, with the following Dirac brackets assuring the validity of the orthonormality condition $\eta^{\mu\nu} - b_{\check{A}}^\mu \eta^{\check{A}\check{b}} b_{\check{B}}^\nu = 0$ [$C_{\gamma\delta}^{\mu\nu\alpha\beta} = \eta_\gamma^\nu\eta_\delta^\alpha\eta^{\mu\beta} + \eta_\gamma^\mu\eta_\delta^\beta\eta^{\nu\alpha} - \eta_\gamma^\nu\eta_\delta^\beta\eta^{\mu\alpha} - \eta_\gamma^\mu\eta_\delta^\alpha\eta^{\nu\beta}$ are the structure constants of the Lorentz group]

$$\{S_s^{\mu\nu}, b_{\check{A}}^\rho\}^* = \eta^{\rho\nu}b_{\check{A}}^\mu - \eta^{\rho\mu}b_{\check{A}}^\nu$$
$$\{S_s^{\mu\nu}, S_s^{\alpha\beta}\}^* = C_{\gamma\delta}^{\mu\nu\alpha\beta} S_s^{\gamma\delta}, \tag{2.27}$$

so that, while $\tilde{\mathcal{H}}^\mu(\tau) \approx 0$ has zero Dirac bracket with itself and with $\tilde{\mathcal{H}}^{\mu\nu}(\tau) \approx 0$, these last six constraints have the Dirac brackets

$$\{\tilde{\mathcal{H}}^{\mu\nu}(\tau), \tilde{\mathcal{H}}^{\alpha\beta}(\tau)\}^* = C_{\gamma\delta}^{\mu\nu\alpha\beta}\tilde{\mathcal{H}}^{\gamma\delta}(\tau) \approx 0. \tag{2.28}$$

We have now only the variables: x_s^μ, p_s^μ, $b_{\check{A}}^\mu$, $S_s^{\mu\nu}$, α^i, Π_i with the following Dirac brackets:

$$\{x_s^\mu(\tau), p_s^\nu(\tau)\}^* = -\eta^{\mu\nu},$$
$$\{S_s^{\mu\nu}(\tau), b_{\check{A}}^\rho(\tau)\}^* = \eta^{\rho\nu}b_{\check{A}}^\mu(\tau) - \eta^{\rho\mu}b_{\check{A}}^\nu(\tau),$$
$$\{S_s^{\mu\nu}(\tau), S_s^{\alpha\beta}(\tau)\}^* = C_{\gamma\delta}^{\mu\nu\alpha\beta} S_s^{\gamma\delta}(\tau),$$
$$\{\alpha^i(\tau,\vec{\sigma}), \Pi_j(\tau,\vec{\sigma}')\}^* = \delta_j^i \delta^3(\vec{\sigma} - \vec{\sigma}'). \tag{2.29}$$

After the restriction to spacelike hyperplanes we have $z_{\check{r}}^\mu(\tau,\vec{\sigma}) \approx b_{\check{r}}^\mu(\tau)$, so that $z_\tau^\mu(\tau,\vec{\sigma}) \approx N_{[z](flat)}(\tau,\vec{\sigma})l^\mu(\tau,\vec{\sigma}) + N_{[z](flat)}^{\check{r}}(\tau,\vec{\sigma}) b_{\check{r}}^{(\mu)}(\tau,\vec{\sigma}) \approx \dot{x}_s^\mu(\tau) + \dot{b}_{\check{r}}^\mu(\tau)\sigma^{\check{r}} = -\tilde{\lambda}^\mu(\tau) - \tilde{\lambda}^{\mu\nu}(\tau)b_{\check{r}\nu}(\tau)\sigma^{\check{r}}$.

As said in the Introduction only now we get the coincidence of the two definitions of flat lapse and shift functions (this point was missed in the older treatments of parametrized Minkowski theories):

$$N_{[z](flat)}(\tau, \vec{\sigma}) \approx N_{(flat)}(\tau, \vec{\sigma}) = -\tilde{\lambda}_\mu(\tau)l^\mu - l^\mu\tilde{\lambda}_{\mu\nu}(\tau)b_{\check{s}}^\nu(\tau)\sigma^{\check{s}} = N(\tau, \vec{\sigma}),$$
$$N_{[z](flat)\check{r}}(\tau, \vec{\sigma}) \approx N_{(flat)\check{r}}(\tau, \vec{\sigma}) = -\tilde{\lambda}_\mu(\tau)b_{\check{r}}^\mu(\tau) - b_{\check{r}}^\mu(\tau)\tilde{\lambda}_{\mu\nu}(\tau)b_{\check{s}}^\nu(\tau)\sigma^{\check{s}} = N_{\check{r}}(\tau, \vec{\sigma}).$$
(2.30)

Let us now restrict ourselves to configurations with $\epsilon p_s^2 > 0$ and let us use the Wigner boost $L^\mu{}_\nu(\overset{\circ}{p}_s, p_s)$ to boost to rest the variables $b_{\check{A}}^\mu$, $S_s^{\mu\nu}$ of the following non-Darboux basis

$$x_s^\mu, p_s^\mu, b_{\check{A}}^\mu, S_s^{\mu\nu}, \alpha^i, \Pi_i$$

of the Dirac brackets $\{.,.\}^*$. The following new non-Darboux basis is obtained [$\tilde{x}_s^\mu$ is no more a fourvector; we choose the sign $\eta = sign\, p_s^o$ positive]

$$\tilde{x}_s^\mu = x_s^\mu + \frac{1}{2}\,\epsilon_\nu^A(u(p_s))\eta_{AB}\frac{\partial\epsilon_\rho^B(u(p_s))}{\partial p_{s\mu}}\,S_s^{\nu\rho} =$$

$$= x_s^\mu - \frac{1}{\sqrt{\epsilon p_s^2}(p_s^o + \eta\sqrt{p_s^2})}\,[p_{s\nu}S_s^{\nu\mu} + \sqrt{\epsilon p_s^2}(S_s^{o\mu} - S_s^{o\nu}\frac{p_{s\nu}p_s^\mu}{\epsilon p_s^2})] =$$

$$= x_s^\mu - \frac{1}{\sqrt{\epsilon p_s^2}}[\eta_A^\mu(\bar{S}_s^{\bar{o}A} - \frac{\bar{S}_s^{Ar}p_s^r}{p_s^o + \sqrt{\epsilon p_s^2}}) + \frac{p_s^\mu + 2\sqrt{\epsilon p_s^2}\eta^{\mu o}}{\sqrt{\epsilon p_s^2}(p_s^o + \sqrt{\epsilon p_s^2})}\bar{S}_s^{\bar{o}r}\,p_s^r],$$

$$p_s^\mu = p_s^\mu,$$

$$\alpha^i = \alpha^i,$$
$$\Pi_i = \Pi_i,$$

$$b_{\check{r}}^A = \epsilon_\mu^A(u(p_s))b_{\check{r}}^\mu,$$
$$\tilde{S}_s^{\mu\nu} = S_s^{\mu\nu} - \frac{1}{2}\epsilon_\rho^A(u(p_s))\eta_{AB}\left(\frac{\partial\epsilon_\sigma^B(u(p_s))}{\partial p_{s\mu}}p_s^\nu - \frac{\partial\epsilon_\sigma^B(u(p_s))}{\partial p_{s\nu}}p_s^\mu\right)S_s^{\rho\sigma} =$$

$$= S_s^{\mu\nu} + \frac{1}{\sqrt{\epsilon p_s^2}(p_s^o + \sqrt{\epsilon p_s^2})}[p_{s\beta}(S_s^{\beta\mu}p_s^\nu - S_s^{\beta\nu}p_s^\mu) + \sqrt{\epsilon p_s^2}(S_s^{o\mu}p_s^\nu - S_s^{o\nu}p_s^\mu)],$$

$$J_s^{\mu\nu} = \tilde{x}_s^\mu p_s^\nu - \tilde{x}_s^\nu p_s^\mu + \tilde{S}_s^{\mu\nu}.$$
(2.31)

We have

$$\{\tilde{x}_s^\mu, p_s^\nu\}^* = 0,$$
$$\{\tilde{S}_s^{oi}, b_{\check{A}}^r\}^* = \frac{\delta^{is}(p_s^r b_{\check{A}}^s - p_s^s b_{\check{A}}^r)}{p_s^o + \sqrt{\epsilon p_s^2}},$$
$$\{\tilde{S}_s^{ij}, b_{\check{A}}^r\}^* = (\delta^{ir}\delta^{js} - \delta^{is}\delta^{jr})b_{\check{A}}^s,$$
$$\{\tilde{S}_s^{\mu\nu}, \tilde{S}_s^{\alpha\beta}\}^* = C_{\gamma\delta}^{\mu\nu\alpha\beta}\tilde{S}_s^{\gamma\delta},$$
(2.32)

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and we can define

$$\bar{S}_s^{AB} = \epsilon_\mu^A(u(p_s))\epsilon_\nu^B(u(p_s))S_s^{\mu\nu} \approx [b_{\check{r}}^A(\tau)b_\tau^B - b_{\check{r}}^B(\tau)b_\tau^A]$$
$$\int d^3\sigma\,\sigma^{\check{r}}\left(J^\tau\sqrt{\mu^2 + \delta_{\check{u}\check{v}}(T^{-1})^{\check{u}i}\Pi_i(T^{-1})^{\check{v}j}\Pi_j}\right)(\tau,\vec{\sigma}) -$$
$$- [b_{\check{r}}^A(\tau)b_{\check{s}}^B(\tau) - b_{\check{r}}^B(\tau)b_{\check{s}}^A(\tau)]\int d^3\sigma\,\sigma^{\check{r}}\left(J^\tau(T^{-1})^{\check{s}l}\Pi_l\right)(\tau,\vec{\sigma}). \tag{2.33}$$

Let us now add six more gauge-fixings by selecting the special family of spacelike hyperplanes $\Sigma_{\tau W}$ orthogonal to p_s^μ (this is possible for $\epsilon p_s^2 > 0$), which can be called the 'Wigner foliation' of Minkowski spacetime. This can be done by requiring (only six conditions are independent)

$$T_{\check{A}}^\mu(\tau) = b_{\check{A}}^\mu(\tau) - \epsilon_{A=\check{A}}^\mu(u(p_s)) \approx 0$$
$$\Rightarrow \quad b_{\check{A}}^A(\tau) = \epsilon_\mu^A(u(p_s))b_{\check{A}}^\mu(\tau) \approx \eta_{\check{A}}^A. \tag{2.34}$$

Now the inverse tetrad $b_{\check{A}}^\mu$ is equal to the polarization vectors $\epsilon_A^\mu(u(p_s))$ [see Appendix C] and the indices '$\check{r}$' are forced to coincide with the Wigner spin-1 indices 'r', while $\bar{o} = \tau$ is a Lorentz-scalar index. One has

$$\bar{S}_s^{AB} \approx (\eta_r^A\eta_\tau^B - \eta_r^B\eta_\tau^A)\bar{S}_s^{\tau r} -$$
$$- (\eta_r^A\eta_s^B - \eta_r^B\eta_s^A)\bar{S}_s^{rs},$$

$$\bar{S}_s^{rs} \approx \int d^3\sigma\left(J^\tau\left[\sigma^r(T^{-1})^{sl}\Pi_l - \sigma^s(T^{-1})^{rl}\Pi_l\right]\right)(\tau,\vec{\sigma}),$$
$$\bar{S}_s^{\tau r} \approx -\bar{S}_s^{r\tau} = -\int d^3\sigma\left(J^\tau\,\sigma^r\sqrt{\mu^2 + \delta_{uv}(T^{-1})^{ui}\Pi_i(T^{-1})^{vj}\Pi_j}\right)(\tau,\vec{\sigma}). \tag{2.35}$$

The comparison of $\bar{S}_s^{AB}$ with $\tilde{S}_s^{\mu\nu}$ yields

$$\tilde{S}_s^{uv} = \delta^{ur}\delta^{vt}\bar{S}_s^{rt}$$
$$\tilde{S}_s^{ov} = -\frac{\delta^{vr}\bar{S}_s^{rt}p_s t}{p_s^0 + \sqrt{\epsilon p_s^2}}. \tag{2.36}$$

The time constancy of $T_{\check{A}}^\mu \approx 0$ with respect to the Dirac Hamiltonian of Eq.(2.24) gives

$$\frac{d}{d\tau}[b_{\check{r}}^\mu(\tau) - \epsilon_r^\mu(u(p_s))] = \{b_{\check{r}}^\mu(\tau) - \epsilon_r^\mu(u(p_s)), H_D\}^* =$$
$$= \frac{1}{2}\tilde{\lambda}^{\alpha\beta}(\tau)\{b_{\check{r}}^\mu(\tau), S_{s\alpha\beta}(\tau)\}^* = \tilde{\lambda}^{\mu\alpha}(\tau)b_{\check{r}\alpha}(\tau) \approx 0$$
$$\Rightarrow \tilde{\lambda}^{\mu\nu}(\tau) \approx 0, \tag{2.37}$$

so that the independent gauge-fixings contained in Eqs.(2.34) and the constraints $\tilde{\mathcal{H}}^{\mu\nu}(\tau) \approx 0$ form six pairs of second class constraints.

Besides Eqs.(2.19), now we have [remember that $\dot{x}_s^\mu(\tau) = -\tilde{\lambda}^\mu(\tau)$]

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$$l^\mu = b^\mu_\tau = u^\mu(p_s),$$

$$z^\mu_\tau(\tau) = \dot{x}^\mu_s(\tau) = \sqrt{g(\tau)}u^\mu(p_s) - \dot{x}_{s\nu}(\tau)\epsilon^\mu_r(u(p_s))\epsilon^\nu_r(u(p_s)),$$

$$N(\tau) = \sqrt{g(\tau)} = [\dot{x}_{s\mu}(\tau)u^\mu(p_s))],\qquad \sqrt{\gamma} = 1,$$

$$g_{\tau\tau} = \dot{x}^2_s,\qquad g_{rs} = -\epsilon^3 g_{rs} = -\epsilon\delta_{rs},$$

$$g_{\tau r} = -\epsilon\dot{x}_{s\mu}\epsilon^\mu_r(u(p_s)) = -\epsilon\delta_{rs}N^s,\qquad N^r = \delta^{ru}\dot{x}_{s\mu}\epsilon^\mu_u(u(p_s)),$$

$$g^{\tau\tau} = \frac{1}{g} = \frac{\epsilon}{N^2},\qquad g^{\tau r} = -\frac{\epsilon}{g}\dot{x}_{s\mu}\delta^{ru}\epsilon^\mu_r(u(p_s)) = -\epsilon\frac{N^r}{N^2},$$

$$g^{rs} = -\epsilon(\delta^{rs} - \delta^{ru}\delta^{sv}\frac{\dot{x}_{s\mu}\epsilon^\mu_u(u(p_s))\dot{x}_{s\nu}\epsilon^\nu_v(u(p_s))}{[\dot{x}_s\cdot u(p_s)]^2}) = -\epsilon(\delta^{rs} - \frac{N^r N^s}{N^2}).\qquad (2.38)$$

On the hyperplane $\Sigma_{\tau W}$ all the degrees of freedom $z^\mu(\tau,\vec{\sigma})$ are reduced to the four degrees of freedom $\tilde{x}^\mu_s(\tau)$, which replace x^μ_s. The Dirac Hamiltonian is now $H_D = \tilde{\lambda}^\mu(\tau)\tilde{\mathcal{H}}_\mu(\tau)$ with

$$\tilde{\mathcal{H}}^\mu(\tau) = p^\mu_s -$$
$$- \int d^3\sigma\Big(J^\tau\Big[u^\mu(p_s)\sqrt{\mu^2 + \delta_{uv}(T^{-1})^{ui}\Pi_i(T^{-1})^{vj}\Pi_j} -$$
$$- \epsilon^\mu_r(u(p_s))\,\mu(T^{-1})^{rl}\Pi_l\Big]\Big)(\tau,\vec{\sigma}) \approx 0.\qquad (2.39)$$

To find the new Dirac brackets, one needs to evaluate the matrix of the old Dirac brackets of the second class constraints (without extracting the independent ones)

$$C = \begin{pmatrix} \{\tilde{\mathcal{H}}^{\alpha\beta},\tilde{\mathcal{H}}^{\gamma\delta}\}^* \approx 0 & \{\tilde{\mathcal{H}}^{\alpha\beta}, T^\sigma_{\check{B}}\}^* = \\ & = \delta_{\check{B}B}[\eta^{\sigma\beta}\epsilon^\alpha_B(u(p_s)) - \eta^{\sigma\alpha}\epsilon^\beta_B(u(p_s))] \\ \{T^\rho_{\check{A}},\tilde{\mathcal{H}}^{\gamma\delta}\}^* = & \{T^\rho_{\check{A}}, T^\sigma_{\check{B}}\}^* = 0 \\ = \delta_{\check{A}A}[\eta^{\rho\gamma}\epsilon^\delta_A(u(p_s)) - \eta^{\rho\delta}\epsilon^\gamma_A(u(p_s))] & . \end{pmatrix}\qquad (2.40)$$

Since the constraints are redundant, this matrix has the following left and right null eigenvectors: $\begin{pmatrix} a_{\alpha\beta} = a_{\beta\alpha} \\ 0 \end{pmatrix}$ $[a_{\alpha\beta}$ arbitrary$]$, $\begin{pmatrix} 0 \\ \epsilon^B_\sigma(u(p_s)) \end{pmatrix}$. Therefore, one has to find a left and right quasi-inverse $\bar{C}$, $\bar{C}C = C\bar{C} = D$, such that $\bar{C}$ and D have the same left and right null eigenvectors. One finds

$$\bar{C} = \begin{pmatrix} 0_{\gamma\delta\mu\nu} & \frac{1}{4}[\eta_{\gamma\tau}\epsilon^D_\delta(u(p_s)) - \eta_{\delta\tau}\epsilon^D_\gamma(u(p_s))] \\ \frac{1}{4}[\eta_{\sigma\nu}\epsilon^B_\mu(u(p_s)) - \eta_{\sigma\mu}\epsilon^B_\nu(u(p_s))] & 0^{BD}_{\sigma\tau} \end{pmatrix}$$

$$\bar{C}C = C\bar{C} = D = \begin{pmatrix} \frac{1}{2}(\eta^\alpha_\mu\eta^\beta_\nu - \eta^\alpha_\nu\eta^\beta_\mu) & 0^{\alpha\beta D}_\tau \\ 0^\rho_{A\mu\nu} & \frac{1}{2}(\eta^\rho_\tau\eta^D_A - \epsilon^{D\rho}(u(p_s))\epsilon_{A\tau}(u(p_s))) \end{pmatrix}\qquad (2.41)$$

and the new Dirac brackets are

$$\{A,B\}^{**} = \{A,B\}^* - \frac{1}{4}[\{A,\tilde{\mathcal{H}}^{\gamma\delta}\}^*[\eta_{\gamma\tau}\epsilon^D_\delta(u(p_s)) - \eta_{\delta\tau}\epsilon^D_\gamma(u(p_s))]\{T^\tau_D, B\}^* +$$
$$+ \{A, T^\sigma_B\}^*[\eta_{\sigma\nu}\epsilon^B_\mu(u(p_s)) - \eta_{\sigma\mu}\epsilon^B_\nu(u(p_s))]\{\tilde{\mathcal{H}}^{\mu\nu}, B\}^*].\qquad (2.42)$$

While the check of $\{\tilde{\mathcal{H}}^{\alpha\beta}, B\}^{**} = 0$ is immediate, we must use the relation $b_{\check{A}\mu}T^\mu_D\epsilon^{D\rho} = -T^\rho_{\check{A}}$ [at this level we have $T^\mu_{\check{A}} = T^\mu_A$] to check $\{T^\rho_A, B\}^{**} = 0$.

Then, we find the following brackets for the remaining variables $\tilde{x}_s^\mu, p_s^\mu, \alpha^i, \Pi_i$

$$\{\tilde{x}_s^\mu, p_s^\nu\}^{**} = -\eta^{\mu\nu},$$
$$\{\alpha^i(\tau, \vec{\sigma}), \Pi_j(\tau, \vec{\sigma}')\}^{**} = \delta_j^i \delta^3(\vec{\sigma} - \vec{\sigma}'), \tag{2.43}$$

and the following form of the generators of the "external" Poincaré group

$$p_s^\mu,$$
$$J_s^{\mu\nu} = \tilde{x}_s^\mu p_s^\nu - \tilde{x}_s^\nu p_s^\mu + \tilde{S}_s^{\mu\nu},$$
$$\tilde{S}_s^{oi} = -\frac{\delta^{ir} \bar{S}_s^{rs} p_s^s}{p_s^o + \sqrt{\epsilon p_s^2}},$$
$$\tilde{S}_s^{ij} = \delta^{ir}\delta^{js}\bar{S}_s^{rs}. \tag{2.44}$$

Let us come back to the four first class constraints $\tilde{\mathcal{H}}^\mu(\tau) \approx 0$, $\{\tilde{\mathcal{H}}^\mu, \tilde{\mathcal{H}}^\nu\}^{**} = 0$, of Eq.(2.25). They can be rewritten in the following form [from Eqs.(1.9), (2.6) we have $J^\tau = -det\,(\partial_r\alpha^i)$, $(T^{-1})^{ri} = \delta^{rs}\partial_s\alpha^i/det\,(\partial_u\alpha^k)$]

$$\mathcal{H}(\tau) = u^\mu(p_s)\tilde{\mathcal{H}}_\mu(\tau) = \epsilon_s - M_{sys} \approx 0,$$

$$M_{sys} = \int d^3\sigma \mathcal{M}(\tau, \vec{\sigma}) =$$
$$= \int d^3\sigma \Big(J^\tau \sqrt{\mu^2 + \delta_{uv}(T^{-1})^{ui}\Pi_i(T^{-1})^{vj}\Pi_j}\Big)(\tau, \vec{\sigma}) =$$
$$= -\int d^3\sigma \Big[det\,(\partial_r\alpha^k) \sqrt{\mu^2 + \delta^{uv}\frac{\partial_u\alpha^i\partial_v\alpha^j}{[det\,(\partial_r\alpha^k)]^2}\Pi_i\Pi_j}\Big](\tau, \vec{\sigma}),$$

$$\vec{\mathcal{H}}_p(\tau) \stackrel{def}{=} \vec{P}_{sys} = \int d^3\sigma \mathcal{M}^r(\tau, \vec{\sigma}) =$$
$$= \int d^3\sigma \mu \Big(J^\tau (T^{-1})^{rl}\Pi_l\Big)(\tau, \vec{\sigma}) = -\int d^3\sigma \, \mu \Big[\delta^{rs}\partial_s\alpha^i\Pi_i\Big](\tau, \vec{\sigma}) \approx 0, \tag{2.45}$$

where M_{sys} is the invariant mass of the fluid. The first one gives the mass spectrum of the isolated system, while the other three say that the total 3-momentum of the N particles on the hyperplane $\Sigma_{\tau W}$ vanishes.

There is no more a restriction on p_s^μ in this special gauge, because $u^\mu(p_s) = p_s^\mu/\sqrt{\epsilon p_s^2}$ gives the orientation of the Wigner hyperplanes containing the isolated system with respect to an arbitrary given external observer. Now the lapse and shift functions are

$$N = N_{[z](flat)} = N_{(flat)} = -\lambda(\tau) = \dot{x}_s^\mu(\tau)u_\mu(p_s),$$
$$N_r = N_{[z](flat)r} = N_{(flat)r} = -\lambda_r(\tau) = -\dot{x}_s^\mu(\tau)\epsilon_{r\mu}(u(p_s)), \tag{2.46}$$

so that the velocity of the origin of the coordinates on the Wigner hyperplane is

$$\dot{x}_s^\mu(\tau) = \epsilon[-\lambda(\tau)u^\mu(p_s) + \lambda_r(\tau)\epsilon_r^\mu(u(p_s)), \quad [u^2(p_s) = \epsilon, \epsilon_r^2(u(p_s)) = -\epsilon]. \tag{2.47}$$

The Dirac Hamiltonian is now

$$H_D = \lambda(\tau)\mathcal{H}(\tau) - \vec{\lambda}(\tau) \cdot \vec{\mathcal{H}}_p(\tau), \tag{2.48}$$

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and we have $\dot{\tilde{x}}^\mu_s = \{\tilde{x}^\mu_s, H_D\}^{**} = -\lambda(\tau)u^\mu(p_s)$. Therefore, while the old x^μ_s had a velocity $\dot{x}^\mu_s$ not parallel to the normal $l^\mu = u^\mu(p_s)$ to the hyperplane as shown by Eqs.(2.47), the new $\tilde{x}^\mu_s$ has $\dot{\tilde{x}}^\mu_s \| l^\mu$ and no classical zitterbewegung. Moreover, we have that $T_s = l \cdot \tilde{x}_s = l \cdot x_s$ is the Lorentz-invariant rest frame time.

The canonical variables $\tilde{x}^\mu_s$, p^μ_s, may be replaced by the canonical pairs $\epsilon_s = \sqrt{p^2_s}$, $T_s = p_s \cdot \tilde{x}_s/\epsilon_s$ [to be gauge fixed with $T_s - \tau \approx 0$]; $\vec{k}_s = \vec{p}_s/\epsilon_s = \vec{u}(p_s)$, $\vec{z}_s = \epsilon_s(\vec{\tilde{x}}_s - \frac{\vec{p}_s}{p^o_s}\tilde{x}^o_s) \equiv \epsilon_s\vec{q}_s$.

One obtains in this way a new kind of instant form of the dynamics, the "Wigner-covariant 1-time rest-frame instant form" with a universal breaking of Lorentz covariance. It is the special relativistic generalization of the non-relativistic separation of the center of mass from the relative motion $[H = \frac{\vec{P}^2}{2M} + H_{rel}]$. The role of the "external" center of mass is taken by the Wigner hyperplane, identified by the point $\tilde{x}^\mu_s(\tau)$ and by its normal p^μ_s. The invariant mass M_{sys} of the system replaces the non-relativistic Hamiltonian H_{rel} for the relative degrees of freedom, after the addition of the gauge-fixing $T_s - \tau \approx 0$ [identifying the time parameter τ, labelling the leaves of the foliation, with the Lorentz scalar time of the "external" center of mass in the rest frame, $T_s = p_s \cdot \tilde{x}_s/M_{sys}$ and implying $\lambda(\tau) = -\epsilon$]. After this gauge fixing the Dirac Hamiltonian would be pure gauge: $H_D = -\vec{\lambda}(\tau) \cdot \vec{\mathcal{H}}_p(\tau)$. However, if we wish to reintroduce the evolution in the time $\tau \equiv T_s$ in this frozen phase space we must use the Hamiltonian [in it the time evolution is generated by M_{sys}: it is like in the frozen Hamilton-Jacobi theory, in which the evolution can be reintroduced by using the energy generator of the Poincaré group as Hamiltonian]

$$H_D = M_{sys} - \vec{\lambda}(\tau) \cdot \vec{\mathcal{H}}_p(\tau). \tag{2.49}$$

The Hamilton equations for $\alpha^i(\tau, \vec{\sigma})$ in the Wigner covariant rest-frame instant form are equivalent to the hydrodynamical Euler equations:

$$\partial_\tau \alpha^i(\tau, \vec{\sigma}) = \{\alpha^i(\tau, \vec{\sigma}), H_D\} =$$

$$= -\left(\frac{\delta^{uv}\partial_u\alpha^i\partial_v\alpha^j\Pi_j}{det\,(\partial_r\alpha^k)\sqrt{\mu^2 + \delta^{uv}\frac{\partial_u\alpha^m\partial_v\alpha^n}{[det\,(\partial_r\alpha^k)]^2}\Pi_m\Pi_n}}\right)(\tau, \vec{\sigma}) +$$

$$+ \lambda^r(\tau)\partial_r\alpha^i(\tau),$$

$$\partial_\tau \Pi_i(\tau, \vec{\sigma}) = \{\Pi_i(\tau, \vec{\sigma}), H_d\} = \qquad (ijk\,cyclic)$$

$$= \frac{\partial}{\partial\sigma^s}\Big[\epsilon^{suv}\partial_u\alpha^j\partial_v\alpha^k\sqrt{\mu^2 + \delta^{uv}\frac{\partial_u\alpha^m\partial_v\alpha^n}{[det\,(\partial_r\alpha^k)]^2}\Pi_m\Pi_n} +$$

$$+ \frac{(\delta^m_i\delta^u_s - \frac{\delta^{uv}\partial_v\alpha^m}{det\,(\partial_r\alpha^l)}\epsilon^{slt}\epsilon_{ipq}\partial_l\alpha^p\partial_t\alpha^q)\frac{\partial_u\alpha^n}{det\,(\partial_r\alpha^l)}\Pi_m\Pi_n}{\sqrt{\mu^2 + \delta^{uv}\frac{\partial_u\alpha^m\partial_v\alpha^n}{[det\,(\partial_r\alpha^k)]^2}\Pi_m\Pi_n}}\Big](\tau, \vec{\sigma}) +$$

$$+ \lambda^r(\tau)\frac{\partial}{\partial\sigma^s}\Big[\epsilon^{suv}\partial_u\alpha^j\partial_v\alpha^k\frac{\partial_r\alpha^m}{det\,(\partial_t\alpha^l)}\Pi_m +$$

$$+ (\delta^m_i\delta_{rs} - \frac{\partial_r\alpha^m}{det\,(\partial_t\alpha^l)}\epsilon^{slt}\epsilon_{ipq}\partial_l\alpha^p\partial_t\alpha^q)\Pi_m\Big](\tau, \vec{\sigma}). \tag{2.50}$$

In this special gauge we have $b^\mu_A \equiv L^\mu{}_A(p_s, \overset{\circ}{p}_s)$ (the standard Wigner boost for timelike Poincaré orbits), $S^{\mu\nu}_s \equiv S^{\mu\nu}_{sys}$ $[S^r_{sys} = \epsilon^{ruv}\int d^3\sigma\, \sigma^u\left(J^\tau\,(T^{-1})^{vs}\Pi_s\right)(\tau, \vec{\sigma})]$, and the only remaining canonical variables are the non-covariant Newton-Wigner-like canonical "external"

3-center-of-mass coordinate $\vec{z}_s$ (living on the Wigner hyperplanes) and $\vec{k}_s$. Now 3 degrees of freedom of the isolated system [an "internal" center-of-mass 3-variable $\vec{q}_{sys}$ defined inside the Wigner hyperplane and conjugate to $\vec{P}_{sys}$] become gauge variables [the natural gauge fixing to the rest-frame condition $\vec{P}_{sys} \approx 0$ is $\vec{X}_{sys} \approx 0$, implying $\lambda_r(\tau) = 0$, so that it coincides with the origin $x_s^\mu(\tau) = z^\mu(\tau, \vec{\sigma} = 0)$ of the Wigner hyperplane]. The variable $\tilde{x}_s^\mu$ is playing the role of a kinematical "external" center of mass for the isolated system and may be interpreted as a decoupled observer with his parametrized clock (point particle clock). All the fields living on the Wigner hyperplane are now either Lorentz scalar or with their 3-indices transformaing under Wigner rotations (induced by Lorentz transformations in Minkowski spacetime) as any Wigner spin 1 index.

III. EXTERNAL AND INTERNAL CANONICAL CENTER OF MASS, MOLLER'S CENTER OF ENERGY AND FOKKER-PRYCE CENTER OF INERTIA

Let us now consider the problem of the definition of the relativistic center of mass of a perfect fluid configuration, using the dust as an example. Let us remark that in the approach leading to the rest-frame instant form of dynamics on Wigner's hyperplanes there is a splitting of this concept in an "external" and an "internal" one. One can either look at the isolated system from an arbitrary Lorentz frame or put himself inside the Wigner hyperplane.

From outside one finds after the canonical reduction to Wigner hyperplane that there is an origin $x_s^\mu(\tau)$ for these hyperplanes (a covariant non-canonical centroid) and a non-covariant canonical coordinate $\tilde{x}_s^\mu(\tau)$ describing an "external" decoupled point particle observer with a clock measuring the rest-frame time T_s. Associated with them there is the "external" realization (2.44) of the Poincaré group.

Instead, all the degrees of freedom of the isolated system (here the perfect fluid configuration) are described by canonical variables on the Wigner hyperplane restricted by the rest-frame condition $\vec{P}_{sys} \approx 0$, implying that an "internal" collective variable $\vec{q}_{sys}$ is a gauge variable and that only relative variables are physical degrees of freedom (a form of weak Mach principle).

Inside the Wigner hyperplane at $\tau = 0$ there is another realization of the Poincaré group, the "internal" Poincaré group. Its generators are built by using the invariant mass M_{sys} and the 3-momentum $\vec{P}_{sys}$, determined by the constraints (2.25), as the generators of the translations and by using the spin tensor $\bar{S}_s^{AB}$ as the generator of the Lorentz subalgebra

$$P^\tau = M_{sys} = \int d^3\sigma \Big[J^\tau \sqrt{\mu^2 + \delta_{uv}(T^{-1})^{ui}(T^{-1})^{vj}\Pi_i\Pi_j} \Big](\tau,\vec{\sigma}),$$

$$P^r = \vec{P}_{sys} = -\int d^3\sigma \Big[J^\tau (T^{-1})^{ri}\Pi_i \Big](\tau,\vec{\sigma}) \approx 0,$$

$$K^r = J^{\tau r} = \bar{S}_s^{\tau r} \equiv \int d^3\sigma \sigma^r \Big[J^\tau \sqrt{\mu^2 + \delta_{uv}(T^{-1})^{ui}(T^{-1})^{vj}\Pi_i\Pi_j} \Big](\tau,\vec{\sigma}),$$

$$J^r = S_{sys}^r = \frac{1}{2}\epsilon^{ruv}\bar{S}_s^{uv} \equiv \epsilon^{ruv}\int d^3\sigma \sigma^u \Big[J^\tau (T^{-1})^{vi}\Pi_i \Big](\tau,\vec{\sigma}). \tag{3.1}$$

By using the methods of Ref. [13] (where there is a complete discussion of many definitions of relativistic center-of-mass-like variables) we can build the three "internal" (that is inside the Wigner hyperplane) Wigner 3-vectors corresponding to the 3-vectors 'canonical center of mass' $\vec{q}_{sys}$, 'Moller center of energy' $\vec{r}_{sys}$ and 'Fokker-Pryce center of inertia' $\vec{y}_{sys}$ [the analogous concepts for the Klein-Gordon field are in Ref. [14] (based on Refs. [15]), while for the relativistic N-body problem see Ref. [16] and for the system of N charged scalar particles plus the electromagnetic field Ref. [17]].

The non-canonical "internal" Møller 3-center of energy and the associated spin 3-vector are

$$\vec{r}_{sys} = -\frac{\vec{K}}{P^\tau} = -\frac{1}{2P^\tau} \int d^3\sigma \, \vec{\sigma} \left[J^\tau \sqrt{\mu^2 + \delta_{uv}(T^{-1})^{ui}(T^{-1})^{vj}\Pi_i\Pi_j} \right](\tau,\vec{\sigma}),$$

$$\vec{\Omega}_{sys} = \vec{J} - \vec{r}_{sys} \times \vec{P},$$

$$\{r^r_{sys}, P^s\} = \delta^{rs}, \qquad \{r^r_{sys}, P^\tau\} = \frac{P^r}{P^\tau},$$

$$\{r^r_{sys}, r^s_{sys}\} = -\frac{1}{(P^\tau)^2}\epsilon^{rsu}\Omega^u_{sys},$$

$$\{\Omega^r_{sys}, \Omega^s_{sys}\} = \epsilon^{rsu}(\Omega^u_{sys} - \frac{1}{(P^\tau)^2}(\vec{\Omega}_{sys} \cdot \vec{P})\ P^u), \qquad \{\Omega^r_{sys}, P^\tau\} = 0. \qquad (3.2)$$

The canonical "internal" 3-center of mass $\vec{q}_{sys}$ [$\{q^r_{sys}, q^s_{sys}\} = 0$, $\{q^r_{sys}, P^s\} = \delta^{rs}$, $\{J^r, q^s_{sys}\} = \epsilon^{rsu}q^u_{sys}$] is

$$\vec{q}_{sys} = \vec{r}_{sys} - \frac{\vec{J} \times \vec{\Omega}_{sys}}{\sqrt{(P^\tau)^2 - \vec{P}^2}(P^\tau + \sqrt{(P^\tau)^2 - \vec{P}^2})} =$$

$$= -\frac{\vec{K}}{\sqrt{(P^\tau)^2 - \vec{P}^2}} + \frac{\vec{J} \times \vec{P}}{\sqrt{(P^\tau)^2 - \vec{P}^2}(P^\tau + \sqrt{(P^\tau)^2 - \vec{P}^2})} +$$

$$+ \frac{(\vec{K} \cdot \vec{P})\ \vec{P}}{P^\tau\sqrt{(P^\tau)^2 - \vec{P}^2}\left(P^\tau + \sqrt{(P^\tau)^2 - \vec{P}^2}\right)},$$

$$\approx \vec{r}_{sys} \quad for \quad \vec{P} \approx 0; \qquad \{\vec{q}_{sys}, P^\tau\} = \frac{\vec{P}}{P^\tau} \approx 0,$$

$$\vec{S}_{q\,sys} = \vec{J} - \vec{q}_{sys} \times \vec{P} =$$

$$= \frac{P^\tau\vec{J}}{\sqrt{(P^\tau)^2 - \vec{P}^2}} + \frac{\vec{K} \times \vec{P}}{\sqrt{(P^\tau)^2 - \vec{P}^2}} - \frac{(\vec{J} \cdot \vec{P})\ \vec{P}}{\sqrt{(P^\tau)^2 - \vec{P}^2}\left(P^\tau + \sqrt{(P^\tau)^2 - \vec{P}^2}\right)} \approx$$

$$\approx \vec{S}_{sys}, \quad for \quad \vec{P} \approx 0, \quad S^r_{sys} = \epsilon^{ruv}\int d^3\sigma\, \sigma^u\left(J^\tau\,(T^{-1})^{vs}\Pi_s\right)(\tau, \vec{\sigma}),$$

$$\{\vec{S}_{q\,sys}, \vec{P}\} = \{\vec{S}_{q\,sys}, \vec{q}_{sys}\} = 0, \qquad \{S^r_{q\,sys}, S^s_{q\,sys}\} = \epsilon^{rsu}S^u_{q\,sys}. \qquad (3.3)$$

The "internal" non-canonical Fokker-Pryce 3-center of inertia' $\vec{y}_{sys}$ is

$$\vec{y}_{sys} = \vec{q}_{sys} + \frac{\vec{S}_{sys} \times \vec{P}}{\sqrt{(P^\tau)^2 - \vec{P}^2}(P^\tau + \sqrt{(P^\tau)^2 - \vec{P}^2})} = \vec{r}_{sys} + \frac{\vec{S}_{sys} \times \vec{P}}{P^\tau\sqrt{(P^\tau)^2 - \vec{P}^2}},$$

$$\vec{q}_{sys} = \vec{r}_{sys} + \frac{\vec{S}_{sys} \times \vec{P}}{P^\tau(P^\tau + \sqrt{(P^\tau)^2 - \vec{P}^2})} = \frac{P^\tau\vec{r}_{sys} + \sqrt{(P^\tau)^2 - \vec{P}^2}\vec{y}_{sys}}{P^\tau + \sqrt{(P^\tau)^2 - \vec{P}^2}},$$

$$\{y^r_{sys}, y^s_{sys}\} = \frac{1}{P^\tau\sqrt{(P^\tau)^2 - \vec{P}^2}}\epsilon^{rsu}\left[S^u_{sys} + \frac{(\vec{S}_{sys} \cdot \vec{P})\ P^u}{\sqrt{(P^\tau)^2 - \vec{P}^2}(P^\tau + \sqrt{(P^\tau)^2 - \vec{P}^2})}\right],$$

$$\vec{P} \approx 0 \Rightarrow \vec{q}_{sys} \approx \vec{r}_{sys} \approx \vec{y}_{sys}. \qquad (3.4)$$

The Wigner 3-vector $\vec{q}_{sys}$ is therefore the canonical 3-center of mass of the perfect fluid configuration [since $\vec{q}_{sys} \approx \vec{r}_{sys}$, it also describe that point $z^\mu(\tau, \vec{q}_{sys}) = x^\mu_s(\tau) + q^r_{sys}\epsilon^\mu_r(u(p_s))$ where the energy of the configuration is concentrated].

There should exist a canonical transformation from the canonical basis $\alpha^i(\tau,\vec{\sigma})$, $\Pi_i(\tau,\vec{\sigma})$, to a new basis $\vec{q}_{sys}$, $\vec{P} = \vec{P}_\phi$, $\alpha^i_{rel}(\tau,\vec{\sigma})$, $\Pi_{rel\,i}(\tau,\vec{\sigma})$ containing relative variables $\alpha^i_{rel}(\tau,\vec{\sigma})$, $\Pi_{rel\,i}(\tau,\vec{\sigma})$ with respect to the true center of mass of the perfect fluid configuration. To identify this final canonical basis one shall need the methods of Ref. [16].

The gauge fixing $\vec{q}_{sys} \approx 0$ [it implies $\vec{\lambda}(\tau) = 0$] forces all three internal center-of-mass variables to coincide with the origin x^μ_s of the Wigner hyperplane. We shall denote $x^{(\vec{q}_{sys})\mu}_s(\tau) = x^\mu_s(0) + \tau u^\mu(p_s)$ the origin in this gauge (it is a special centroid among the many possible ones; $x^\mu_s(0)$ is arbitrary).

As we shall see in the next Section, by adding the gauge fixings $\vec{X}_{sys} = \vec{q}_{sys} \approx 0$ one can show that the origin $x^\mu_s(\tau)$ becomes simultaneously the Dixon center of mass of an extended object and both the Pirani and Tulczyjew centroids (see Ref. [18] for a review of these concepts in relation with the Papapetrou-Dixon-Souriau pole-dipole approximation of an extended body). The worldline $x^{(\vec{q}_{sys})\mu}_s$ is the unique center-of-mass worldline of special relativity in the sense of Refs. [19].

With similar methods from the rest-frame instant form "external" realization of the Poincaré algebra of Eq. (2.44) with the generators p^μ_s, $J^{ij}_s = \tilde{x}^i_s p^j_s - \tilde{x}^j_s p^i_s + \delta^{ir}\delta^{js}S^{rs}_\phi$, $K^i_s = J^{oi}_s = \tilde{x}^o_s p^i_s - \tilde{x}^i_s p^o_s - \frac{\delta^{ir}S^{rs}_\phi p^s_s}{p^o_s + \epsilon_s} = \tilde{x}^o_s p^i_s - \tilde{x}^i_s p^o_s + \delta^{ir}\frac{(\vec{S}_\phi \times \vec{p}_s)^r}{p^o_s + \epsilon_s}$ [for $\tilde{x}^o_s = 0$ this is the Newton-Wigner decomposition of $J^{\mu\nu}_s$] we can build three "external" collective 3-positions (all located on the Wigner hyperplane): i) the "external canonical 3-center of mass $\vec{Q}_s$ connected with the "external" canonical non-covariant center of mass $\tilde{x}^\mu_s$; ii) the "external" Møller 3-center of energy $\vec{R}_s$ connected with the "external" non-canonical and non-covariant Møller center of energy R^μ_s; iii) the "external" Fokker-Pryce 3-center of inertia connected with the "external" covariant non-canonical Fokker-Price center of inertia Y^μ_s (when there are the gauge fixings $\vec{\sigma}_{sys} \approx 0$ it coincides with the origin x^μ_s). It turns out that the Wigner hyperplane is the natural setting for the study of the Dixon multipoles of extended relativistic systems [20] (see next Section) and for defining the canonical relative variables with respect to the center of mass.

The three "external" 3-variables, the canonical $\vec{Q}_s$, the Møller $\vec{R}_s$ and the Fokker-Pryce $\vec{Y}_s$ built by using the rest-frame "external" realization of the Poincaré algebra are

$$\vec{R}_s = -\frac{1}{p^o_s}\vec{K}_s = (\tilde{\vec{x}}_s - \frac{\vec{p}_s}{p^o_s}\tilde{x}^o_s) - \frac{\vec{S}_{sys} \times \vec{p}_s}{p^o_s(p^o_s + \epsilon_s)},$$

$$\vec{Q}_s = \tilde{\vec{x}}_s - \frac{\vec{p}_s}{p^o_s}\tilde{x}^o_s = \frac{\vec{z}_s}{\epsilon_s} = \vec{R}_s + \frac{\vec{S}_{sys} \times \vec{p}_s}{p^o_s(p^o_s + \epsilon_s)} = \frac{p^o_s \vec{R}_s + \epsilon_s \vec{Y}_s}{p^o_s + \epsilon_s},$$

$$\vec{Y}_s = \vec{Q}_s + \frac{\vec{S}_{sys} \times \vec{p}_s}{\epsilon_s(p^o_s + \epsilon_s)} = \vec{R}_s + \frac{\vec{S}_{sys} \times \vec{p}_s}{p^o_s \epsilon_s},$$

$$\{R^r_s, R^s_s\} = -\frac{1}{(p^o_s)^2}\epsilon^{rsu}\Omega^u_s, \qquad \vec{\Omega}_s = \vec{J}_s - \vec{R}_s \times \vec{p}_s,$$

$$\{Y^r_s, Y^s_s\} = \frac{1}{\epsilon_s p^o_s}\epsilon^{rsu}\Big[S^u_{sys} + \frac{(\vec{S}_{sys} \cdot \vec{p}_s)\,p^u_s}{\epsilon_s(p^o_s + \epsilon_s)}\Big],$$

$$\vec{p}_s \cdot \vec{Q}_s = \vec{p}_s \cdot \vec{R}_s = \vec{p}_s \cdot \vec{Y}_s = \vec{k}_s \cdot \vec{z}_s,$$

$$\vec{p}_s = 0 \Rightarrow \vec{Q}_s = \vec{Y}_s = \vec{R}_s, \tag{3.5}$$

with the same velocity and coinciding in the Lorentz rest frame where $\overset{\circ}{p}{}_s^\mu = \epsilon_s(1; \vec{0})$

In Ref. [13] in a one-time framework without constraints and at a fixed time, it is shown that the 3-vector $\vec{Y}_s$ [but not $\vec{Q}_s$ and $\vec{R}_s$] satisfies the condition $\{K_s^r, Y_s^s\} = Y_s^r \{Y_s^s, p_s^o\}$ for being the space component of a 4-vector Y_s^μ. In the enlarged canonical treatment including time variables, it is not clear which are the time components to be added to $\vec{Q}_s$, $\vec{R}_s$, $\vec{Y}_s$, to rebuild 4-dimesnional quantities $\tilde{x}_s^\mu$, R_s^μ, Y_s^μ, in an arbitrary Lorentz frame Γ, in which the origin of the Wigner hyperplane is the 4-vector $x_s^\mu = (x_s^o; \vec{x}_s)$. We have

$$\tilde{x}_s^\mu(\tau) = (\tilde{x}_s^o(\tau); \vec{\tilde{x}}_s(\tau)) = x_s^\mu - \frac{1}{\epsilon_s(p_s^o + \epsilon_s)}\Big[p_{s\nu}S_s^{\nu\mu} + \epsilon_s(S_s^{o\mu} - S_s^{o\nu}\frac{p_{s\nu}p_s^\mu}{\epsilon_s^2})\Big], \qquad p_s^\mu,$$

$$\tilde{x}_s^o = \sqrt{1 + \vec{k}_s^2}(T_s + \frac{\vec{k}_s \cdot \vec{z}_s}{\epsilon_s}) = \sqrt{1 + \vec{k}_s^2}(T_s + \vec{k}_s \cdot \vec{q}_s) \neq x_s^0, \qquad p_s^o = \epsilon_s\sqrt{1 + \vec{k}_s^2},$$

$$\vec{\tilde{x}}_s = \frac{\vec{z}_s}{\epsilon_s} + (T_s + \frac{\vec{k}_s \cdot \vec{z}_s}{\epsilon_s})\vec{k}_s = \vec{q}_s + (T_s + \vec{k}_s \cdot \vec{q}_s)\vec{k}_s, \qquad \vec{p}_s = \epsilon_s\vec{k}_s. \tag{3.6}$$

for the non-covariant (frame-dependent) canonical center of mass and its conjugate momentum.

Each Wigner hyperplane intersects the worldline of the arbitrary origin 4-vector $x_s^\mu(\tau) = z^\mu(\tau, \vec{0})$ in $\vec{\sigma} = 0$, the pseudo worldline of $\tilde{x}_s^\mu(\tau) = z^\mu(\tau, \vec{\tilde{\sigma}})$ in some $\vec{\tilde{\sigma}}$ and the worldline of the Fokker-Pryce 4-vector $Y_s^\mu(\tau) = z^\mu(\tau, \vec{\sigma}_Y)$ in some $\vec{\sigma}_Y$ [on this worldline one can put the "internal center of mass" with the gauge fixing $\vec{q}_\phi \approx 0$ ($\vec{q}_\phi \approx \vec{r}_\phi \approx \vec{y}_\phi$ due to $\vec{P}_\phi \approx 0$)]; one also has $R_s^\mu = z^\mu(\tau, \vec{\sigma}_R)$. Since we have $T_s = u(p_s) \cdot x_s = u(p_s) \cdot \tilde{x}_s \equiv \tau$ on the Wigner hyperplane labelled by τ, we require that also Y_s^μ, R_s^μ have time components such that they too satisfy $u(p_s) \cdot Y_s = u(p_s) \cdot R_s = T_s \equiv \tau$. Therefore, it is reasonable to assume that $\tilde{x}_s^\mu$, Y_s^μ and R_s^μ satisfy the following equations consistently with Eqs.(3.2), (3.3) when $T_s \equiv \tau$ and $\vec{q}_{sys} \approx 0$

$$\tilde{x}_s^\mu = (\tilde{x}_s^o; \vec{\tilde{x}}_s) = (\tilde{x}_s^o; \vec{Q}_s + \frac{\vec{p}_s}{p_s^o}\tilde{x}_s^o) =$$

$$= (\tilde{x}_s^o; \frac{\vec{z}_s}{\epsilon_s} + (T_s + \frac{\vec{k}_s \cdot \vec{z}_s}{\epsilon_s})\vec{k}_s) = x_s^{(\vec{q}_{sys})\mu} + \epsilon_u^\mu(u(p_s))\tilde{\sigma}^u,$$

$$Y_s^\mu = (\tilde{x}_s^o; \vec{Y}_s) =$$

$$= (\tilde{x}_s^o; \frac{1}{\epsilon_s}[\vec{z}_s + \frac{\vec{S}_{sys} \times \vec{p}_s}{\epsilon_s[1 + u^o(p_s)]}] + (T_s + \frac{\vec{k}_s \cdot \vec{z}_s}{\epsilon_s})\vec{k}_s) =$$

$$= \tilde{x}_s^\mu + \eta_r^\mu \frac{(\vec{S}_{sys} \times \vec{p}_s)^r}{\epsilon_s[1 + u^o(p_s)]} =$$

$$= x_s^{(\vec{q}_{sys})\mu} + \epsilon_u^\mu(u(p_s))\sigma_Y^u,$$

$$R_s^\mu = (\tilde{x}_s^o; \vec{R}_s) =$$

$$= (\tilde{x}_s^o; \frac{1}{\epsilon_s}[\vec{z}_s - \frac{\vec{S}_{sys} \times \vec{p}_s}{\epsilon_s u^o(p_s)[1 + u^o(p_s)]}] + (T_s + \frac{\vec{k}_s \cdot \vec{z}_s}{\epsilon_s})\vec{k}_s) =$$

$$= \tilde{x}_s^\mu - \eta_r^\mu \frac{(\vec{S}_{sys} \times \vec{p}_s)^r}{\epsilon_s u^o(p_s)[1 + u^o(p_s)]} =$$

$$= x_s^{(\vec{q}_{sys})\mu} + \epsilon_u^\mu(u(p_s))\sigma_R^u,$$

$$T_s = u(p_s) \cdot x_s^{(\vec{q}_{sys})} = u(p_s) \cdot \tilde{x}_s = u(p_s) \cdot Y_s = u(p_s) \cdot R_s,$$

$$\tilde{\sigma}^r = \epsilon_{r\mu}(u(p_s))[x_s^{(\vec{q}_{sys})\mu} - \tilde{x}_s^\mu] = \frac{\epsilon_{r\mu}(u(p_s))[u_\nu(p_s)S_s^{\nu\mu} + S_s^{o\mu}]}{[1 + u^o(p_s)]} =$$

$$= -S_{sys}^{\tau r} + \frac{S_{sys}^{rs}p_s^s}{\epsilon_s[1 + u^o(p_s)]} = \epsilon_s r_\phi^r + \frac{S_{sys}^{rs}u^s(p_s)}{1 + u^o(p_s)} \approx$$

$$\approx \epsilon_s q_{sys}^r + \frac{S_{sys}^{rs}u^s(p_s)}{1 + u^o(p_s)} \approx \frac{S_{sys}^{rs}u^s(p_s)}{1 + u^o(p_s)},$$

$$\sigma_Y^r = \epsilon_{r\mu}(u(p_s))[x_s^{(\vec{q}_{sys})\mu} - Y_s^\mu] = \tilde{\sigma}^r - \epsilon_{ru}(u(p_s))\frac{(\vec{S}_{sys} \times \vec{p}_s)^u}{\epsilon_s[1 + u^o(p_s)]} =$$

$$= \tilde{\sigma}^r + \frac{S_{sys}^{rs}u^s(p_s)}{1 + u^o(p_s)} = \epsilon_s r_{sys}^r \approx \epsilon_s q_{sys}^r \approx 0,$$

$$\sigma_R^r = \epsilon_{r\mu}(u(p_s))[x_s^{(\vec{q}_{sys})\mu} - R_s^\mu] = \tilde{\sigma}^r + \epsilon_{ru}(u(p_s))\frac{(\vec{S}_{sys} \times \vec{p}_s)^u}{\epsilon_s u^o(p_s)[1 + u^o(p_s)]} =$$

$$= \tilde{\sigma}^r - \frac{S_{sys}^{rs}u^s(p_s)}{u^o(p_s)[1 + u^o(p_s)]} = \epsilon_s r_{sys}^r + \frac{[1 - u^o(p_s)]S_{sys}^{rs}u^s(p_s)}{u^o(p_s)[1 + u^o(p_s)]} \approx$$

$$\approx \frac{[1 - u^o(p_s)]S_{sys}^{rs}u^s(p_s)}{u^o(p_s)[1 + u^o(p_s)]},$$

$$\Rightarrow x_s^{(\vec{q}_{sys})\mu}(\tau) = Y_s^\mu, \quad for \quad \vec{q}_{sys} \approx 0, \tag{3.7}$$

namely in the gauge $\vec{q}_{sys} \approx 0$ the external Fokker-Pryce non-canonical center of inertia coincides with the origin $x_s^{(\vec{q}_{sys})\mu}(\tau)$ carrying the "internal" center of mass (coinciding with the "internal" Möller center of energy and with the "internal" Fokker-Pryce center of inertia) and also being the Pirani centroid and the Tulczyjew centroid.

Therefore, if we would find the center-of-mass canonical basis, then, in the gauge $\vec{q}_{sys} \approx 0$ and $T_s \approx \tau$, the perfect fluid configurations would have the four-momentum density peaked on the worldline $x_s^{(\vec{q}_{sys})\mu}(T_s)$; the canonical variables $\alpha_{rel}^i(\tau, \vec{\sigma})$, $\Pi_{rel\,i}(\tau, \vec{\sigma})$ would characterize the relative motions with respect to the "monopole" configuration describing the center of mass of the fluid configuration. The "monopole" configurations would be identified by the vanishing of the relative variables.

Remember that the canonical center of mass lies in between the Moller center of energy and the Fokker-Pryce center of inertia and that the non-covariance region around the Fokker-Pryce 4-vector extends to a worldtube with radius (the Moller radius) $|\vec{S}_{sys}|/P^\tau$.

IV. DIXON'S MULTIPOLES IN MINKOWSKI SPACETIME.

Let us now look at other properties of a perfect fluid configuration on the Wigner hyperplanes, always using the dust as an explicit example. To identify which kind of collective variables describe the center of mass of a fluid configuration let us consider it as a relativistic extended body and let us study its energy-momentum tensor and its Dixon multipoles [20] in Minkowski spacetime.

The Euler-Lagrange equations from the action (1.10) [(2.1) for the dust] are

$$\Big(\frac{\partial \mathcal{L}}{\partial z^\mu} - \partial_{\check{A}}\frac{\partial \mathcal{L}}{\partial z^\mu_{\check{A}}}\Big)(\tau,\vec{\sigma}) = \eta_{\mu\nu}\partial_{\check{A}}[\sqrt{g}T^{\check{A}\check{B}}[\alpha]z^\nu_{\check{B}}](\tau,\vec{\sigma}) \overset{\circ}{=} 0,$$

$$\Big(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_{\check{A}}\frac{\partial \mathcal{L}}{\partial \partial_{\check{A}}\phi}\Big)(\tau,\vec{\sigma}) \overset{\circ}{=} 0, \tag{4.1}$$

where we introduced the energy-momentum tensor [with a different sign with respect to the standard convention to conform with Ref. [1]]

$$T^{\check{A}\check{B}}(\tau,\vec{\sigma})[\alpha] = \Big[\frac{2}{\sqrt{g}}\frac{\delta S}{\delta g_{\check{A}\check{B}}}\Big](\tau,\vec{\sigma}) =$$

$$= \Big[- \rho g^{\check{A}\check{B}} + n\frac{\partial \rho}{\partial n}|_s\,(g^{\check{A}\check{B}} - \frac{J^{\check{A}}J^{\check{B}}}{g_{\check{C}\check{D}}J^{\check{C}}J^{\check{D}}})\Big](\tau,\vec{\sigma}) =$$

$$= \Big[- \epsilon\rho U^{\check{A}}U^{\check{B}} + p(g^{\check{A}\check{B}} - \epsilon U^{\check{A}}U^{\check{B}})\Big](\tau,\vec{\sigma})$$

$$\overset{dust}{=} -\epsilon\mu n U^{\check{A}}U^{\check{B}} = -\epsilon\frac{J^{\check{A}}J^{\check{B}}}{N^2\sqrt{\gamma}\,J^\tau}\sqrt{\mu^2 + {}^3g_{\check{u}\check{v}}(T^{-1})^{\check{u}i}(T^{-1})^{\check{v}j}\Pi_i\Pi_j}. \tag{4.2}$$

When $\partial_{\check{A}}[\sqrt{g}z^\mu_{\check{B}}] = 0$, as it happens on the Wigner hyperplanes in the gauge $T_s - \tau \approx 0$, $\vec{\lambda}(\tau) = 0$, we get the conservation of the energy-momentum tensor $T^{\check{A}\check{B}}$, i.e. $\partial_{\check{A}}T^{\check{A}\check{B}} \overset{\circ}{=} 0$. Otherwise, there is compensation coming from the dynamics of the surface.

As shown in Eq.(A9) the conserved, manifestly Lorentz covariant energy-momentum tensor of the perfect fluid with equation of state $\rho = \rho(n,s)$ [so that $p = n\frac{\partial \rho}{\partial n}|_s - \rho)$] is

$$T^{\mu\nu}(x)[\tilde{\alpha}] = \Big[- \epsilon\rho\,{}^4g^{\mu\nu} + n\frac{\partial \rho}{\partial n}|_s\,({}^4g^{\mu\nu} - \frac{J^\mu J^\nu}{{}^4g_{\alpha\beta}J^\alpha J^\beta})\Big](x) =$$

$$= \Big[- \epsilon\rho U^\mu U^\nu + p({}^4g^{\mu\nu} - \epsilon U^\mu U^\nu)\Big](x) =$$

$$= \Big[- \epsilon(\rho + p)U^\mu U^\nu + p\,{}^4g^{\mu\nu}\Big](x)$$

$$\overset{dust}{=} -\epsilon\mu\Big[n\,U^\mu U^\nu\Big](x),$$

$$nU^\mu = J^\mu = -\epsilon^{\mu\nu\rho\sigma}\partial_\nu\tilde{\alpha}^1\partial_\rho\tilde{\alpha}^2\partial_\sigma\tilde{\alpha}^3 = nz^\mu_{\check{A}}U^{\check{A}} =$$

$$= z^\mu_{\check{A}}J^{\check{A}} = z^\mu_\tau J^\tau + z^\mu_{\check{r}}J^{\check{r}}. \tag{4.3}$$

Therefore, in Σ_τ-adapted coordinates on each Σ_τ we get

$$T^{\check{A}\check{B}}(\tau,\vec{\sigma})[\alpha] = z_\mu^{\check{A}}(\tau,\vec{\sigma})z_\nu^{\check{B}}(\tau,\vec{\sigma})T^{\mu\nu}(x=z(\tau,\vec{\sigma}))[\tilde{\alpha}] =$$
$$= z_\mu^{\check{A}}(\tau,\vec{\sigma})z_\nu^{\check{B}}(\tau,\vec{\sigma})T^{\mu\nu}(\tau,\vec{\sigma})[\alpha = \tilde{\alpha}\circ z], \tag{4.4}$$

On Wigner hyperplanes, where Eqs.(2.38) hold and where we have

$$z^\mu(\tau,\vec{\sigma}) = x_s^\mu(\tau) + \epsilon_u^\mu(u(p_s))\sigma^u,$$

$$^4\eta^{\mu\nu} = \epsilon\Big[u^\mu(p_s)u^\nu(p_s) - \sum_{r=1}^3 \epsilon_r^\mu(u(p_s))\epsilon_r^\nu(u(p_s))\Big],$$

$$N = \dot{x}_s \cdot u(p_s), \qquad N^r = \delta^{ru}\dot{x}_{s\mu}\epsilon_u^\mu(u(p_s)),$$

$$Y^r = \frac{J^r + N^r J^\tau}{N} = \frac{J^r + \delta^{ru}\dot{x}_{s\mu}\epsilon_u^\mu(u(p_s))J^\tau}{\dot{x}_s \cdot u(p_s)},$$

$$n = \frac{\sqrt{\epsilon g_{AB}J^A J^B}}{N} = \sqrt{(J^\tau)^2 - \delta_{rs}Y^r Y^s}$$

$$\overset{dust}{=} \frac{\mu J^\tau}{\sqrt{\mu^2 + \delta_{uv}(T^{-1})^{ui}(T^{-1})^{vj}\Pi_i\Pi_j}}, \tag{4.5}$$

we get $[\check{A} = A]$

$$T^{\mu\nu}[x_s^\beta(\tau) + \epsilon_u^\beta(u(p_s))\sigma^u][\alpha] =$$

$$= [\delta_A^\tau \dot{x}_s^\mu(\tau) + \delta_A^r \epsilon_r^\mu(u(p_s))][\delta_B^\tau \dot{x}_s^\nu(\tau) + \delta_B^s \epsilon_s^\nu(u(p_s))]T^{AB}(\tau,\vec{\sigma}) =$$
$$= \dot{x}_s^\mu(\tau)\dot{x}_s^\nu(\tau)T^{\tau\tau}(\tau,\vec{\sigma}) + \epsilon_r^\mu(u(p_s))\epsilon_s^\nu(u(p_s))T^{rs}(\tau,\vec{\sigma}) +$$
$$+ [\dot{x}_s^\mu(\tau)\epsilon_r^\nu(u(p_s)) + \dot{x}_s^\nu(\tau)\epsilon_r^\mu(u(p_s))]T^{r\tau}(\tau,\vec{\sigma}),$$

$$T^{\tau\tau}(\tau\vec{\sigma}) = \Big[-\frac{\epsilon\rho}{[\dot{x}_s \cdot u(p_s)]^2} +$$
$$+ n\frac{\partial\rho}{\partial n}\Big|_s \Big(\frac{\epsilon}{[\dot{x}_s \cdot u(p_s)]^2} - \frac{(J^\tau)^2}{[\dot{x}_s \cdot u(p_s)]^2[(J^\tau)^2 - \delta_{uv}Y^u Y^v]}\Big)\Big](\tau,\vec{\sigma})$$
$$\overset{dust}{=} -\epsilon\Big[\frac{J^\tau}{[\dot{x}_s \cdot u(p_s)]^2}\sqrt{\mu^2 + \delta_{uv}(T^{-1})^{ui}(T^{-1})^{vj}\Pi_i\Pi_j}\Big](\tau,\vec{\sigma}),$$

$$T^{r\tau}(\tau,\vec{\sigma}) = T^{\tau r}(\tau,\vec{\sigma}) = \Big[-\epsilon\rho\frac{\delta^{ru}\dot{x}_{s\mu}\epsilon_u^\mu(u(p_s))}{[\dot{x}_s \cdot u(p_s)]^2} -$$
$$- n\frac{\partial\rho}{\partial n}\Big|_s \Big(\epsilon\frac{\delta^{ru}\dot{x}_{s\mu}\epsilon_u^\mu(u(p_s))}{[\dot{x}_s \cdot u(p_s)]^2} +$$
$$+ \frac{J^r J^\tau}{[\dot{x}_s \cdot u(p_s)]^2[(J^\tau)^2 - \delta_{uv}Y^u Y^v]}\Big)\Big](\tau,\vec{\sigma})$$
$$\overset{dust}{=} -\epsilon\Big[\frac{J^r}{[\dot{x}_s \cdot u(p_s)]^2}\sqrt{\mu^2 + \delta_{uv}(T^{-1})^{ui}(T^{-1})^{vj}\Pi_i\Pi_j}\Big](\tau,\vec{\sigma}),$$

$$T^{rs}(\tau,\vec{\sigma}) = \Big[\epsilon\rho(\delta^{rs} - \delta^{ru}\delta^{sv}\frac{\dot{x}_{s\mu}\epsilon_u^\mu(u(p_s))\dot{x}_{s\nu}\epsilon_v^\nu(u(p_s))}{[\dot{x}_s \cdot u(p_s)]^2}) -$$

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$$- n\frac{\partial\rho}{\partial n}|_s \left(\epsilon(\delta^{rs} - \delta^{ru}\delta^{sv}\frac{\dot{x}_{s\mu}\epsilon_u^\mu(u(p_s))\dot{x}_{s\nu}\epsilon_v^\nu(u(p_s))}{[\dot{x}_s \cdot u(p_s)]^2}) +$$

$$+ \frac{J^r J^s}{[\dot{x}_s \cdot u(p_s)]^2[\dot{x}_s \cdot u(p_s)]^2[(J^\tau)^2 - \delta_{uv}Y^uY^v]}\right)\Big](\tau, \vec{\sigma})$$

$$\overset{dust}{=} -\epsilon\Big[\frac{J^r J^s}{[\dot{x}_s \cdot u(p_s)]^2 \, J^\tau}\sqrt{\mu^2 + \delta_{uv}(T^{-1})^{ui}(T^{-1})^{vj}\Pi_i\Pi_j}\Big](\tau, \vec{\sigma}). \tag{4.6}$$

Since we have

$$\dot{x}_s^\mu(\tau) = -\lambda^\mu(\tau) = \epsilon[u^\mu(p_s)u^\nu(p_s) - \epsilon_r^\mu(u(p_s))\epsilon_r^\nu(u(p_s))]\dot{x}_{s\nu}(\tau) =$$
$$= \epsilon\Big[-u^\mu(p_s)\lambda(\tau) + \epsilon_r^\mu(u(p_s))\lambda_r(\tau)\Big],$$
$$\dot{x}_s^2(\tau) = \lambda^2(\tau) - \vec{\lambda}(\tau) > 0,$$
$$U_s^\mu(\tau) = \frac{\dot{x}_s^\mu(\tau)}{\sqrt{\dot{x}_s^2(\tau)}} = \epsilon\frac{-\lambda(\tau)u^\mu(p_s) + \lambda_r(\tau)\epsilon_r^\mu(u(p_s))}{\sqrt{\lambda^2(\tau) - \vec{\lambda}^2(\tau)}}, \tag{4.7}$$

the timelike worldline described by the origin of the Wigner hyperplane is arbitrary (i.e. gauge dependent): $x_s^\mu(\tau)$ may be any covariant non-canonical centroid. As already said the real "external" center of mass is the canonical non-covariant $\tilde{x}_s^\mu(T_s) = x_s^\mu(T_s) - \frac{1}{\epsilon_s(p_s^o + \epsilon_s)}\Big[p_{s\nu}S_s^{\nu\mu} + \epsilon_s(S_s^{o\mu} + S_s^{o\nu}\frac{p_{s\nu}p_s^\mu}{\epsilon_s^2})\Big]$: it describes a decoupled point particle observer.

In the gauge $T_s - \tau \approx 0$, $\vec{X}_{sys} = \vec{q}_{sys} \approx 0$, implying $\lambda(\tau) = -1$, $\vec{\lambda}(\tau) = 0$ [$g_{\tau\tau} = \epsilon$, $N = 1$, $N^r = g_{\tau r} = 0$], we get $\dot{x}_s^\mu(T_s) = u^\mu(p_s)$. Therefore, in this gauge, we have the centroid

$$x_s^\mu(T_s) = x_s^{(\vec{q}_{sys})\mu}(T_s) = x_s^\mu(0) + T_s u^\mu(p_s), \tag{4.8}$$

which carries the fluid "internal" collective variable $\vec{X}_{sys} = \vec{q}_{sys} \approx 0$.

In this gauge we get the following form of the energy-momentum tensor [$Y^r = J^r$]

$$T^{\mu\nu}[x_s^{(\vec{q}_{sys})\beta}(T_s) + \epsilon_u^\beta(u(p_s))\sigma^u][\alpha] = u^\mu(p_s)u^\nu(p_s)T^{\tau\tau}(T_s, \vec{\sigma}) +$$
$$+ [u^\mu(p_s)\epsilon_r^\nu(u(p_s)) + u^\nu(p_s)\epsilon_r^\mu(u(p_s))]T^{r\tau}(T_s, \vec{\sigma}) +$$
$$+ \epsilon_r^\mu(u(p_s))\epsilon_s^\nu(u(p_s))T^{rs}(T_s, \vec{\sigma}),$$

$$T^{\tau\tau}(T_s, \vec{\sigma}) = \Big[-\epsilon\rho +$$
$$+ n\frac{\partial\rho}{\partial n}|_s\left(\epsilon - \frac{(J^\tau)^2}{(J^\tau)^2 - \delta_{uv}Y^uY^v}\right)\Big](T_s, \vec{\sigma})$$
$$\overset{dust}{=} -\epsilon\Big[J^\tau\sqrt{\mu^2 + \delta_{uv}(T^{-1})^{ui}(T^{-1})^{vj}\Pi_i\Pi_j}\Big](T_s, \vec{\sigma}),$$

$$T^{r\tau}(T_s, \vec{\sigma}) = \Big[-n\frac{\partial\rho}{\partial n}|_s\frac{J^r J^\tau}{(J^\tau)^2 - \delta_{uv}Y^uY^v}\Big](T_s, \vec{\sigma})$$
$$\overset{dust}{=} -\epsilon\Big[J^r\sqrt{\mu^2 + \delta_{uv}(T^{-1})^{ui}(T^{-1})^{vj}\Pi_i\Pi_j}\Big](T_s, \vec{\sigma}),$$

$$T^{rs}(T_s, \vec{\sigma}) = \Big[\epsilon\rho\delta^{rs} - n\frac{\partial\rho}{\partial n}|_s\left(\epsilon\delta^{rs} +$$
$$+ \frac{J^r J^s}{(J^\tau)^2 - \delta_{uv}Y^uY^v}\right)\Big](T_s, \vec{\sigma})$$

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$$\overset{dust}{=} -\epsilon\Big[\frac{J^r J^s}{J^\tau}\sqrt{\mu^2 + \delta_{uv}(T^{-1})^{ui}(T^{-1})^{vj}\Pi_i\Pi_j}\Big](T_s, \vec\sigma),$$

$$with \ total \ 4 - momentum$$

$$P_T^\mu[\alpha] = \int d^3\sigma T^{\mu\nu}[x_s^\mu(T_s) + \epsilon_u^\mu(u(p_s))\sigma^u][\alpha]u_\nu(p_s) =$$
$$= -P^\tau u^\mu(p_s) - P^r\epsilon_r^\mu(u(p_s)) \approx -P^\tau u^\mu(p_s) =$$
$$= -M_{sys}u^\mu(p_s) \approx -p_s^\mu,$$

$$and \ total \ mass$$

$$M[\alpha] = P_T^\mu[\alpha]u_\mu(p_s) = -P^\tau = -M_{sys}. \tag{4.9}$$

The stress tensor of the perfect fluid configuration on the Wigner hyperplanes is $T^{rs}(T_s, \vec\sigma)$.

We can rewrite the energy-momentum tensor in such a way that it acquires a form reminiscent of the energy-momentum tensor of an ideal relativistic fluid as seen from a local observer at rest (see the Eckart decomposition in Appendix B):

$$T^{\mu\nu}[\tilde\alpha] = \Big[\rho[\alpha,\Pi] \ u^\mu(p_s)u^\nu(p_s) +$$
$$+ \mathcal{P}[\alpha,\Pi] \ [\eta^{\mu\nu} - u^\mu(p_s)u^\nu(p_s)] +$$
$$+ u^\mu(p_s)q^\nu[\alpha,\Pi] + u^\nu(p_s)q^\mu[\alpha,\Pi] +$$
$$+ T_{an}^{rs}[\alpha,\Pi] \ \epsilon_r^\mu(u(p_s))\epsilon_s^\nu(u(p_s))\Big](T_s, \vec\sigma),$$

$$\rho[\alpha,\Pi] = T^{\tau\tau},$$
$$\mathcal{P}[\alpha,\Pi] = \frac{1}{3}\sum_u T^{uu},$$
$$q^\mu[\alpha,\Pi] = \epsilon_r^\mu(u(p_s))T^{r\tau},$$
$$T_{an}^{rs}[\alpha,\Pi] = T^{rs} - \frac{1}{3}\delta^{rs}\sum_u T^{uu}, \qquad \delta_{uv}T_{an}^{uv}[\alpha,\Pi] = 0, \tag{4.10}$$

where

i) the constant normal $u^\mu(p_s)$ to the Wigner hyperplanes replaces the hydrodynamic velocity field of the fluid;

ii) $\rho[\alpha,\Pi](T_s, \vec\sigma)$ is the energy density;

iii) $\mathcal{P}[\alpha,\Pi](T_s, \vec\sigma)$ is the analogue of the pressure (sum of the thermodynamical pressure and of the non-equilibrium bulk stress or viscous pressure);

iv) $q^\mu[\alpha,\Pi](T_s, \vec\sigma)$ is the analogue of the heat flow;

v) $T_{an}^{rs}[\alpha,\Pi](T_s, \vec\sigma)$ is the shear (or anisotropic) stress tensor.

We can now study the manifestly Lorentz covariant Dixon multipoles [20] for the perfect fluid configurationon the Wigner hyperplanes in the gauge $\lambda(\tau) = -1$, $\vec\lambda(\tau) = 0$ [so that $\dot{x}_s^\mu(T_s) = u^\mu(p_s)$, $\ddot{x}_s^\mu(T_s) = 0$, $x_s^\mu(T_s) = x_s^{(\vec{q}_{sys})\mu}(T_s) = x_s^\mu(0) + u^\mu(p_s)T_s$] with respect to the origin an arbitrary timelike worldline $w^\mu(T_s) = z^\mu(T_s, \vec\eta(T_s)) = x_s^{(\vec{q}_{sys})\mu}(T_s) +$

$\epsilon_r^\mu(u(p_s))\eta^r(T_s)$. Since we have $z^\mu(T_s,\vec\sigma) = x_s^{(\vec q_{sys})\mu}(T_s)+\epsilon_r^\mu(u(p_s))\sigma^r = w^\mu(T_s)+\epsilon_r^\mu(u(p_s))[\sigma^r - \eta^r(T_s)] \overset{def}{=} w^\mu(T_s) + \delta z^\mu(T_s,\vec\sigma)$ [for $\vec\eta(T_s) = 0$ we get the multipoles with respect to the origin of coordinates], we obtain [$(\mu_1..\mu_n)$ means symmetrization, while $[\mu_1..\mu_n]$ means antisymmetrization; $t_T^{\mu_1..\mu_n\mu\nu}(T_s,\vec\eta = 0) = t_T^{\mu_1..\mu_n\mu\nu}(T_s)$]

$$t_T^{\mu_1\cdots\mu_n\mu\nu}(T_s,\vec\eta) = t_T^{(\mu_1\cdots\mu_n)(\mu\nu)}(T_s,\vec\eta) =$$
$$= \int d^3\sigma\, \delta z^{\mu_1}(T_s,\vec\sigma)...\delta z^{\mu_n}(T_s,\vec\sigma)\, T^{\mu\nu}[x_s^{(\vec q_{sys})\beta}(T_s) + \epsilon_u^\beta(u(p_s))\sigma^u][\alpha] =$$
$$= \epsilon_{r_1}^{\mu_1}(u(p_s))...\epsilon_{r_n}^{\mu_n}(u(p_s))\epsilon_A^\mu(u(p_s))\epsilon_B^\nu(u(p_s))\, I_T^{r_1..r_nAB}(T_s,\vec\eta) =$$
$$= \epsilon_{r_1}^{\mu_1}(u(p_s))...\epsilon_{r_n}^{\mu_n}(u(p_s))\Big[u^\mu(p_s)u^\nu(p_s)I_T^{r_1\cdots r_n\tau\tau}(T_s,\vec\eta) +$$
$$+ \epsilon_r^\mu(u(p_s))\epsilon_s^\nu(u(p_s))I_T^{r_1\cdots r_nrs}(T_s,\vec\eta) +$$
$$+ [u^\mu(p_s)\epsilon_r^\nu(u(p_s)) + u^\nu(p_s)\epsilon_r^\mu(u(p_s))]I_T^{r_1\cdots r_nr\tau}(T_s,\vec\eta)\Big],$$

$$I_T^{r_1..r_nAB}(T_s,\vec\eta) = \int d^3\sigma\, [\sigma^{r_1} - \eta^{r_1}(T_s)]...[\sigma^{r_n} - \eta^{r_n}(T_s)]T^{AB}(T_s,\vec\sigma)[\alpha],$$

$$u_{\mu_1}(p_s) \quad t_T^{\mu_1\cdots\mu_n\mu\nu}(T_s,\vec\eta) = 0,$$

$$For\,\vec\eta = 0 \quad n = 0\,(monopole) \quad I_T^{\tau\tau}(T_s) = P^\tau, \qquad I_T^{r\tau}(T_s) = P^r,$$

$$t_T^{\mu_1\cdots\mu_n\mu}{}_\mu(T_s) = \int d^3\sigma\delta x_s^{\mu_1}(\vec\sigma)...\delta x_s^{\mu_n}(\vec\sigma)T^\mu{}_\mu[x_s^{(\vec q_{sys})\beta}(T_s) + \epsilon_u^\beta(u(p_s))\sigma^u][\alpha] =$$
$$\overset{def}{=} \epsilon_{r_1}^{\mu_1}(u(p_s))...\epsilon_{r_n}^{\mu_n}(u(p_s))I_T^{r_1\cdots r_nA}{}_A(T_s)$$

$$I_T^{r_1r_2A}{}_A(T_s) = \check I_T^{r_1r_2A}{}_A(T_s) - \frac{1}{3}\delta^{r_1r_2}\delta_{uv}I_T^{uvA}{}_A(T_s) = i_T^{r_1r_2}(T_s) - \frac{1}{2}\delta^{r_1r_2}\delta_{uv}i_T^{uv}(T_s),$$

$$\check I_T^{r_1r_2A}{}_A(T_s) = i_T^{r_1r_2}(T_s) - \frac{1}{3}\delta^{r_1r_2}\delta_{uv}i_T^{uv}(T_s), \qquad \delta_{uv}\check I_T^{uvA}{}_A(T_s) = 0,$$
$$i_T^{r_1r_2}(T_s) = I_T^{r_1r_2A}{}_A(T_s) - \delta^{r_1r_2}\delta_{uv}I_T^{uvA}{}_A(T_s),$$

$$\tilde t_T^{\mu_1\cdots\mu_n}(T_s) = t_T^{\mu_1\cdots\mu_n\mu\nu}(T_s)u_\mu(p_s)u_\nu(p_s) =$$
$$= \epsilon_{r_1}^{\mu_1}(u(p_s))...\epsilon_{r_n}^{\mu_n}(u(p_s))I_T^{r_1..r_n\tau\tau}(T_s),$$

$$\tilde t_T^{\mu_1}(T_s) = \epsilon_{r_1}^{\mu_1}(u(p_s))I_T^{r_1\tau\tau}(T_s) = -P^\tau\epsilon_{r_1}^{\mu_1}(u(p_s))r_{sys}^{r_1},$$

$$\tilde t_T^{\mu_1\mu_2}(T_s) = \epsilon_{r_1}^{\mu_1}(u(p_s))\epsilon_{r_2}^{\mu_2}(u(p_s))I_T^{r_1r_2\tau\tau}(T_s),$$

$$I_T^{r_1r_2\tau\tau}(T_s) = \hat I_T^{r_1r_2\tau\tau}(T_s) - \frac{1}{3}\delta^{r_1r_2}\delta_{uv}I_T^{uv\tau\tau}(T_s) = \tilde i_T^{r_1r_2}(T_s) - \frac{1}{2}\delta^{r_1r_2}\delta_{uv}\tilde i_T^{uv}(T_s),$$

$$\hat I_T^{r_1r_2\tau\tau}(T_s) = \tilde i_T^{r_1r_2}(T_s) - \frac{1}{3}\delta^{r_1r_2}\delta_{uv}\tilde i_T^{uv}(T_s), \qquad \delta_{uv}\hat I_T^{uv\tau\tau}(T_s) = 0,$$
$$\tilde i_T^{r_1r_2}(T_s) = I_T^{r_1r_2\tau\tau}(T_s) - \delta^{r_1r_2}\delta_{uv}I_T^{uv\tau\tau}(T_s). \tag{4.11}$$

The Wigner covariant multipoles $I_T^{r_1..r_n\tau\tau}(T_s)$, $I_T^{r_1..r_nrs}(T_s)$, $I_T^{r_1..r_nr\tau}(T_s)$ are the mass,

stress and momentum multipoles respectively.

The quantities $\check{I}_T^{r_1 r_2 A}{}_A(T_s)$ and $i_T^{r_1 r_2}(T_s)$ are the traceless quadrupole moment and the inertia tensor defined by Thorne in Ref. [21].

The quantities $I_T^{r_1 r_2 \tau \tau}(T_s)$ and $\tilde{i}_T^{r_1 r_2}(T_s)$ are Dixon's definitions of quadrupole moment and of tensor of inertia respectively.

Moreover, Dixon's definition of "center of mass" of an extended object is $\tilde{t}_T^{\mu_1}(T_s) = 0$ or $I_T^{r \tau \tau}(T_s) = -P^\tau r_{sys}^r = 0$: therefore the quantity $\vec{r}_{sys}$ defined in the previous equation is a non-canonical $[\{r_{sys}^r, r_{sys}^s\} = S_{sys}^{rs}]$ candidate for the "internal" center of mass of the field configuration: its vanishing is a gauge fixing for $\vec{P} \approx 0$ and implies $x_s^\mu(T_s) = x_s^{(\vec{q}_{sys})\mu}(T_s) = x_s^\mu(0) + u^\mu(p_s)T_s$. As we have seen in the previous Section $\vec{r}_{sys}$ is the "internal" Møller 3-center of energy and we have $\vec{r}_{sys} \approx \vec{q}_{sys} \approx \vec{y}_{sys}$.

When $I_T^{r \tau \tau}(T_s) = 0$, the equations $0 = \frac{dI_T^{r \tau \tau}(T_s)}{dT_s} = -P^\tau \frac{dr_{sys}^r}{dT_s} \overset{\circ}{=} \overset{\circ}{=} - P^r$ implies the correct momentum-velocity relation $\frac{\vec{P}}{P^\tau} \overset{\circ}{=} \frac{d\vec{r}_{sys}}{dT_s} \approx 0$.

Then there are the related Dixon multipoles

$$p_T^{\mu_1 \dots \mu_n \mu}(T_s) = t_T^{\mu_1 \dots \mu_n \mu \nu}(T_s)u_\nu(p_s) = p_T^{(\mu_1 \dots \mu_n)\mu}(T_s) =$$
$$= \epsilon_{r_1}^{\mu_1}(u(p_s))\dots\epsilon_{r_n}^{\mu_n}(u(p_s))\epsilon_A^\mu(u(p_s))I_T^{r_1 .. r_n A \tau}(T_s),$$

$$u_{\mu_1}(p_s) \quad p_T^{\mu_1 \dots \mu_n \mu}(T_s) = 0,$$

$$n = 0 \quad \Rightarrow p_T^\mu(T_s) = P_T^\mu[\alpha] = -\epsilon \epsilon_A^\mu(u(p_s))P^A \approx -\epsilon p_s^\mu,$$

$$p_T^{\mu_1 .. \mu_n \mu}(T_s)u_\mu(p_s) = \tilde{t}_T^{\mu_1 \dots \mu_n}(T_s) = \epsilon_{r_1}^{\mu_1}(u(p_s))\dots\epsilon_{r_n}^{\mu_n}(u(p_s))I_T^{r_1 .. r_n \tau \tau}(T_s). \tag{4.12}$$

The spin dipole is defined as

$$S_T^{\mu\nu}(T_s)[\alpha] = 2p_T^{[\mu\nu]}(T_s) = 2\epsilon_r^{[\mu}(u(p_s))\epsilon_A^{\nu]}(u(p_s))I_T^{r A \tau}(T_s) =$$
$$= S_s^{\mu\nu} =$$
$$= \epsilon_r^\mu(u(p_s))\epsilon_s^\nu(u(p_s))S_{sys}^{rs} + [\epsilon_r^\mu(u(p_s))u^\nu(p_s) - \epsilon_r^\nu(u(p_s))u^\mu(p_s)]S_{sys}^{\tau r},$$

$$u_\mu(p_s)S_T^{\mu\nu}(T_s)[\alpha] = -\epsilon_r^\nu(u(p_s))S_{sys}^{\tau r} = -\tilde{t}_T^\nu(T_s) = P^\tau \epsilon_r^\nu(u(p_s))r_{sys}^r, \tag{4.13}$$

with $u_\mu(p_s)S_T^{\mu\nu}(T_s)[\alpha] = 0$ when $\tilde{t}_T^{\mu_1}(T_s) = 0$ and this condition can be taken as a definition of center of mass equivalent to Dixon's one. When this condition holds, the barycentric spin dipole is $S_T^{\mu\nu}(T_s)[\alpha] = 2\epsilon_r^{[\mu}(u(p_s))\epsilon_s^{\nu]}(u(p_s))I_T^{rs\tau}(T_s)$, so that $I_T^{[rs]\tau}(T_s) = \epsilon_\mu^r(u(p_s))\epsilon_\nu^s(u(p_s))S_T^{\mu\nu}(T_s)[\alpha]$.

As shown in Ref. [20], if the fluid configuration has a compact support W on the Wigner hyperplanes $\Sigma_{W\tau}$ and if $f(x)$ is a C^∞ complex-valued scalar function on Minkowski spacetime with compact support [so that its Fourier transform $\tilde{f}(k) = \int d^4x f(x)e^{ik\cdot x}$ is a slowly increasing entire analytic function on Minkowski spacetime $(|(x^o + iy^o)^{q_o}\dots(x^3 + iy^3)^{q_3}f(x^\mu + iy^\mu)| < C_{q_o \dots q_3}e^{a_o|y^o| + \dots + a_3|y^3|}$, $a_\mu > 0$, q_μ positive integers for every μ and $C_{q_o \dots q_3} > 0)$, whose inverse is $f(x) = \int \frac{d^4k}{(2\pi)^4}\tilde{f}(k)e^{-ik\cdot x}$], we have [we consider $\vec{\eta} = 0$ with $\delta z^\mu = \delta x_s^\mu$]

$$< T^{\mu\nu}, f > \; = \int d^4x T^{\mu\nu}(x)f(x) =$$

$$= \int dT_s \int d^3\sigma f(x_s + \delta x_s) T^{\mu\nu}[x_s(T_s) + \delta x_s(\vec{\sigma})][\alpha] =$$

$$= \int dT_s \int d^3\sigma \int \frac{d^4k}{(2\pi)^4} \tilde{f}(k) e^{-ik\cdot[x_s(T_s)+\delta x_s(\vec{\sigma})]} T^{\mu\nu}[x_s(T_s) + \delta x_s(\vec{\sigma})][\alpha] =$$

$$= \int dT_s \int \frac{d^4k}{(2\pi)^4} \tilde{f}(k) e^{-ik\cdot x_s(T_s)} \int d^3\sigma T^{\mu\nu}[x_s(T_s) + \delta x_s(\vec{\sigma})][\alpha]$$

$$\sum_{n=0}^{\infty} \frac{(-i)^n}{n!} [k_\mu \epsilon_u^\mu(u(p_s))\sigma^u]^n =$$

$$= \int dT_s \int \frac{d^4k}{(2\pi)^4} \tilde{f}(k) e^{-ik\cdot x_s(T_s)} \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} k_{\mu_1}...k_{\mu_n} t_T^{\mu_1...\mu_n\mu\nu}(T_s), \qquad (4.14)$$

and, but only for $f(x)$ analytic in W [20], we get

$$< T^{\mu\nu}, f > \; = \int dT_s \sum_{n=0}^{\infty} \frac{1}{n!} t_T^{\mu_1...\mu_n\mu\nu}(T_s) \frac{\partial^n f(x)}{\partial x^{\mu_1}...\partial x^{\mu_n}}\Big|_{x=x_s(T_s)},$$

$$\Downarrow$$

$$T^{\mu\nu}(x)[\alpha] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial x^{\mu_1}...\partial x^{\mu_n}} \int dT_s \delta^4(x - x_s(T_s)) t_T^{\mu_1...\mu_n\mu\nu}(T_s) =$$

$$= \epsilon_A^\mu(u(p_s))\epsilon_B^\nu(u(p_s)) T^{AB}(T_s, \vec{\sigma})[\alpha] =$$

$$= \epsilon_A^\mu(u(p_s))\epsilon_B^\nu(u(p_s)) \sum_{n=0}^{\infty} \frac{(-)^n}{n!} I_T^{r_1...r_n AB}(T_s, \vec{\eta}) \frac{\partial^n \delta^3(\vec{\sigma} - \vec{\eta}(T_s))}{\partial \sigma^{r_1}...\partial \sigma^{r_n}}\Big|_{\vec{\eta}=0}. \qquad (4.15)$$

For a non analytic $f(x)$ we have

$$< T^{\mu\nu}, f > \; = \int dT_s \sum_{n=0}^{N} \frac{1}{n!} t_T^{\mu_1...\mu_n\mu\nu}(T_s) \frac{\partial^n f(x)}{\partial x^{\mu_1}...\partial x^{\mu_n}}\Big|_{x=x_s(T_s)} +$$

$$+ \int dT_s \int \frac{d^4k}{(2\pi)^4} \tilde{f}(k) e^{-ik\cdot x_s(T_s)} \sum_{n=N+1}^{\infty} \frac{(-i)^n}{n!} k_{\mu_1}...k_{\mu_n} t_T^{\mu_1...\mu_n\mu\nu}(T_s), \qquad (4.16)$$

and, as shown in Ref. [20], from the knowledge of the moments $t_T^{\mu_1...\mu_n\mu}(T_s)$ for all $n > N$ we can get $T^{\mu\nu}(x)$ and, thus, all the moments with $n \leq N$.

In Appendix D other types of Dixon's multipoles are analyzed. From this study it turns out that the multipolar expansion(4.15) may be rearranged with the help of the Hamilton equations implying $\partial_\mu T^{\mu\nu} \stackrel{\circ}{=} 0$, so that for analytic fluid configurations from Eq.(D5) we get

$$T^{\mu\nu}(x)[\alpha] \stackrel{\circ}{=} -\epsilon u^{(\mu}(p_s)\epsilon_A^{\nu)}(u(p_s)) \int dT_s \, \delta^4(x - x_s(T_s)) \; P^A +$$

$$+ \frac{1}{2} \frac{\partial}{\partial x^\rho} \int dT_s \, \delta^4(x - x_s(T_s)) \, S_T^{\rho(\mu}(T_s)[\alpha] \; u^{\nu)}(p_s) +$$

$$+ \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial x^{\mu_1}...\partial x^{\mu_n}} \int dT_s \, \delta^4(x - x_s(T_s)) \; \mathcal{I}_T^{\mu_1..\mu_n\mu\nu}(T_s),$$

$$T^{\mu\nu}(w + \delta z) = -\epsilon u^{(\mu}(p_s) \, \epsilon_A^{\nu)}(u(p_s)) P^A \delta^3(\vec{\sigma} - \vec{\eta}(T_s)) +$$

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$$+ \frac{1}{2} S_T^{\rho(\mu}(T_s, \vec{\eta})[\alpha] u^{\nu)}(p_s) \epsilon_\rho^r(u(p_s)) \frac{\partial \delta^3(\vec{\sigma} - \vec{\eta}(t_s))}{\partial \sigma^r} +$$

$$+ \sum_{n=2}^{\infty} \frac{(-)^n}{n!} \Big[\frac{n+3}{n+1} u^\mu(p_s) u^\nu(p_s) I_T^{r_1 \dots r_n \tau \tau}(T_s, \vec{\eta}) +$$

$$+ \frac{1}{n} [u^\mu(p_s) \epsilon_r^\nu(u(p_s)) + u^\nu(p_s) \epsilon_r^\mu(u(p_s))] I_T^{r_1 \dots r_n r \tau}(T_s, \vec{\eta}) +$$

$$+ \epsilon_{s_1}^\mu(u(p_s)) \epsilon_{s_2}^\nu(u(p_s)) [I_T^{r_1 \dots r_n s_1 s_2}(T_s, \vec{\eta}) -$$

$$- \frac{n+1}{n} (I_T^{(r_1 \dots r_n s_1) s_2}(T_s, \vec{\eta}) + I_T^{(r_1 \dots r_n s_2) s_1}(T_s, \vec{\eta})) +$$

$$+ I_T^{(r_1 \dots r_n s_1 s_2)}(T_s, \vec{\eta})] \Big], \tag{4.17}$$

where for $n \geq 2$ and $\vec{\eta} = 0$ $\quad \mathcal{I}_T^{\mu_1 \cdot \cdot \mu_n \mu \nu}(T_s) = \frac{4(n-1)}{n+1} J_T^{(\mu_1 \cdot \cdot \mu_{n-1} |\mu| \mu_n) \nu}(T_s)$, with the quantities $J_T^{\mu_1 \cdot \cdot \mu_n \mu \nu \rho \sigma}(T_s)$ being the Dixon 2^{2+n}-pole inertial moment tensors given in Eqs.(D7) [the quadrupole and related inertia tensor are proportional to $I_T^{r_1 r_2 \tau \tau}(T_s)$].

The equations $\partial_\mu T^{\mu\nu} \overset{\circ}{=} 0$ imply the Papapetrou-Dixon-Souriau equations for the 'pole-dipole' system $P_T^\mu(T_s)$ and $S_T^{\mu\nu}(T_s)[\alpha]$ [see Eqs.(D1) and (D4); here $\vec{\eta} = 0$]

$$\frac{dP_T^\mu(T_s)}{dT_s} \overset{\circ}{=} 0,$$

$$\frac{dS_T^{\mu\nu}(T_s)[\alpha]}{dT_s} \overset{\circ}{=} 2 P_T^{[\mu}(T_s) \ u^{\nu]}(p_s) = -2 \epsilon P^r \epsilon_r^{[\mu}(u(p_s)) \ u^{\nu]}(p_s) \approx 0. \tag{4.18}$$

The Cartesian Dixon's multipoles could be re-expressed in terms of either spherical or STF (symmetric tracefree) multipoles [21] [both kinds of tensors are associated with the irreducible representations of the rotation group: one such multipole of order l has exactly $2l + 1$ independent components].

V. ISENTROPIC AND NON-ISENTROPIC FLUIDS.

Let us now consider isentropic ($s = const.$) perfect fluids. For them we have from Eqs.(1.9), (1.10) and (2.1) [in general μ is not the chemical potential but only a parameter]

$$n = \frac{|J|}{N\sqrt{\gamma}} = \frac{X}{\sqrt{\gamma}},$$

$$\rho = \rho(n) = \rho(\frac{|J|}{N\sqrt{\gamma}}) = \mu f(\frac{X}{\sqrt{\gamma}}),$$

$$L = -\mu N\sqrt{\gamma} f(\frac{X}{\sqrt{\gamma}}). \tag{5.1}$$

Some possible equations of state for such fluids are (see also Appendix A):

1) $p = 0$, dust: this implies

$$\rho(n) = \mu n = \mu \frac{X}{\sqrt{\gamma}}, \qquad i.e.$$

$$f(\frac{X}{\sqrt{\gamma}}) = \frac{X}{\sqrt{\gamma}}, \qquad \frac{\partial f(\frac{X}{\sqrt{\gamma}})}{\partial X} = \frac{1}{\sqrt{\gamma}}. \tag{5.2}$$

2) $p = k\rho(n) = n\frac{\partial \rho(n)}{\partial n} - \rho(n)$ ($k \neq -1$ because otherwise $\rho = const.$, $\mu = 0$, $f = -Ts$). For $k = \frac{1}{3}$ one has the photon gas. The previous differential equation for $\rho(n)$ implies

$$\rho(n) = (an)^{k+1} = \mu n^{k+1} = \mu(\frac{X}{\sqrt{\gamma}})^{k+1}, \qquad (\mu = a^{k+1}), \quad i.e.$$

$$f(\frac{X}{\sqrt{\gamma}}) = (\frac{X}{\sqrt{\gamma}})^{k+1}, \qquad \frac{\partial f(\frac{X}{\sqrt{\gamma}})}{\partial X} = \frac{k+1}{\sqrt{\gamma}}(\frac{X}{\sqrt{\gamma}})^{k}, \tag{5.3}$$

[for $k \to 0$ we recover 1)].

More in general one can have $k = k(s)$: this is a non-isentropic perfect fluid with $\rho = \rho(n, s)$.

3) $p = k\rho^{\gamma}(n) = n\frac{\partial \rho(n)}{\partial n} - \rho(n)$ ($\gamma \neq 1$) [22]. It is an isentropic polytropic perfect fluid ($\gamma = 1 + \frac{1}{n}$). The differential equation for $\rho(n)$ implies [a is an integration constant; the chemical potential is $\mu = \frac{\partial \rho}{\partial n}|_s$]

$$\rho(n) = \frac{an}{[1 - k(an)^{\gamma-1}]^{\frac{1}{\gamma-1}}} = \frac{an}{[1 - k(an)^{\frac{1}{n}}]^{n}}, \qquad i.e.$$

$$f(\frac{X}{\sqrt{\gamma}}) = \frac{\frac{X}{\sqrt{\gamma}}}{[1 - k(a\frac{X}{\sqrt{\gamma}})^{\gamma-1}]^{\frac{1}{\gamma-1}}} = \frac{\frac{X}{\sqrt{\gamma}}}{[1 - k(a\frac{X}{\sqrt{\gamma}})^{\frac{1}{n}}]^{n}},$$

$$\frac{\partial f(\frac{X}{\sqrt{\gamma}})}{\partial X} = \frac{1}{\sqrt{\gamma}}[1 - k(a\frac{X}{\sqrt{\gamma}})^{\gamma-1}]^{-\frac{\gamma}{\gamma-1}} = \frac{1}{\sqrt{\gamma}}[1 - k(a\frac{X}{\sqrt{\gamma}})^{\frac{1}{n}}]^{-(n+1)}. \tag{5.4}$$

Instead in Ref. [23,24] a polytropic perfect fluid is defined by the equation of state [see the last of Eqs.(B4); m is a mass]

$$\rho(n, ns) = mn + \frac{k(s)}{\gamma - 1}(mn)^{\gamma},$$

(5.5)

and has pressure $p = k(s)(mn)^{\gamma} = (\gamma - 1)(\rho - mn)$ and chemical potential or specific enthalpy $\mu' = mc^2 + mk(s)\frac{\gamma}{\gamma - 1}(mn)^{\gamma - 1}$ of Eq.(B3).

4) $p = p(\rho)$, barotropic perfect fluid. In the isentropic case one gets $\rho = \rho(n)$ by solving $p(\rho(n)) = n\frac{\partial \rho(n)}{\partial n} - \rho(n)$.

5) Relativistic ideal (Boltzmann) gas [2] (this is a non-isentropic case):

$$p = nk_B T \quad and \quad \rho = mc^2 n\Gamma(\beta) - p, \qquad \mu = \frac{\rho + p}{n} = mc^2\Gamma(\beta),$$

(5.6)

with $\beta = \frac{mc^2}{k_B T}$, $\Gamma(\beta) = \frac{K_3(\beta)}{K_2(\beta)}$ (K_i are modified Bessel functions). One gets the equation of state $\rho = \rho(n, s)$ by solving the differential equation

$$\frac{\partial \rho}{\partial n}\Big|_s = mc^2\Gamma\left(\frac{mc^2 n}{n\frac{\partial \rho}{\partial n}\big|_s - \rho}\right).$$

(5.7)

5a) Ultrarelativistic case $\beta << 1$ ($mc^2 << k_B T$): since we have $\Gamma(\beta) \approx \frac{4}{\beta} + \frac{\beta}{2} + O(\beta^3)$, we get $\rho = 3nk_B T + \frac{m^2 c^4 n}{2k_B T} + O(k_B T\beta^4)$, namely $p \approx \frac{1}{3}\rho$ and $\rho \approx \rho(n) = \mu n^{4/3}$.

5b) Non-relativistic case $\beta >> 1$ ($k_B T << mc^2$): since we have $\Gamma(\beta) \approx 1 + \frac{5}{2\beta} + O(\beta^{-2})$, we get $\rho \approx mc^2 n + \frac{3}{2}p$, so that we have to solve the differential equation $3n\frac{\partial \rho}{\partial n}\big|_s - 5\rho + 2mc^2 n \approx 0$. Its solution is $\rho(n, s) \approx mc^2 n + k(s)n^{5/3}$. To find $k(s)$ let us use the definition of temperature: $p = nk_B T = k_B\frac{\partial \rho}{\partial s}\big|_n = k_B n^{5/3}\frac{\partial k(s)}{\partial s} = n\frac{\partial \rho}{\partial n}\big|_s - \rho = \frac{2}{3}n^{5/3}k(s)$. This leads to the equation $dln\,k(s) = \frac{2ds}{3k_B}$, whose solution is $k(s) = he^{\frac{2s}{3k_B}} = e^{\frac{2(s-s_o)}{3k_B}}$ with $h = e^{-\frac{2s_o}{3k_B}} = const.$. Therefore, in this case we get [it is a polytropic like in Eq.(5.5) with $\gamma = 5/3$ and $k(s) = \frac{2}{3}m^{-5/3}e^{\frac{2(s-s_o)}{3k_B}}$]

$$\rho(n, s) \approx mc^2 n + n^{\frac{5}{3}}e^{\frac{2(s-s_o)}{3k_B}} \quad and \quad T = \frac{1}{n}\frac{\partial \rho}{\partial s}\Big|_n \approx \frac{2}{3k_B}n^{\frac{5}{3}}e^{\frac{2(s-s_o)}{3k_B}}.$$

(5.8)

The action for these fluids is [$s = s(\alpha^i)$]

$$S = \int d\tau d^3\sigma L(\alpha^i(\tau, \vec{\sigma}), z^{\mu}(\tau, \vec{\sigma})) = -\int d\tau d^3\sigma N\sqrt{\gamma}\rho(n, s),$$

(5.9)

and we have as in Section II

$$J^{\tau} = -\epsilon^{\check{r}\check{u}\check{v}}\partial_{\check{r}}\alpha^1\partial_{\check{u}}\alpha^2\partial_{\check{v}}\alpha^3,$$

$$J^{\check{r}} = \sum_{i=1}^{3}\partial_{\tau}\alpha^i\epsilon^{\check{r}\check{u}\check{v}}\partial_{\check{u}}\alpha^j\partial_{\check{v}}\alpha^k,$$

$$i,\ j,\ k,\ = \text{ cyclic},$$

$$|J| = \sqrt{N^2(J^{\tau})^2 - {}^3g_{\check{r}\check{s}}[J^{\check{r}} + N^{\check{r}}J^{\tau}][J^{\check{v}} + N^{\check{v}}J^{\tau}]} = NX,$$

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$$X = \sqrt{(J^\tau)^2 - {}^3g_{\check{r}\check{s}}Y^{\check{r}}Y^{\check{s}}},$$

$$Y^{\check{r}} = \frac{1}{N}(J^{\check{r}} + N^{\check{r}}J^\tau),$$

$$\frac{\partial X}{\partial \partial_\tau \alpha^i} = \frac{Y^{\check{r}}T_{\check{r}i}}{NX},$$

$$T_{\check{t}i} = -{}^3g_{\check{t}\check{r}}\epsilon^{\check{r}\check{u}\check{v}}\partial_{\check{u}}\alpha^j\partial_{\check{v}}\alpha^k, \quad (i,j,k \quad cyclic),$$

$$\frac{\partial X}{\partial N} = \frac{{}^3g_{\check{r}\check{s}}Y^{\check{r}}Y^{\check{s}}}{NX} = \frac{(J^\tau)^2 - X^2}{NX},$$

$$\frac{\partial X}{\partial N^{\check{u}}} = -J^\tau\frac{{}^3g_{\check{u}\check{s}}Y^{\check{s}}}{NX}. \tag{5.10}$$

In the cases 1), 2) and 3) [in case 3) we rename μ the constant a] the canonical momenta can be written in the form

$$\Pi_i(\tau,\vec{\sigma}) = \frac{\partial L(\tau,\vec{\sigma})}{\partial \partial_\tau \alpha^i(\tau,\vec{\sigma})} = \mu\Big[\frac{\partial f(x)}{\partial x}|_{x=\frac{X}{\sqrt{\gamma}}}\frac{Y^{\check{r}}T_{\check{r}i}}{X}\Big](\tau,\vec{\sigma}),$$

$$\Rightarrow \quad Y^{\check{r}} = -\frac{(T^{-1})^{\check{r}i}\Pi_i}{\mu}X\Big(\frac{\partial f(x)}{\partial x}|_{x=\frac{X}{\sqrt{\gamma}}}\Big)^{-1} = Y^{\check{r}}(X),$$

$$\Rightarrow \quad \partial_\tau \alpha^i = -\frac{J^{\check{r}}\partial_{\check{r}}\alpha^i}{J^\tau} = \frac{N^{\check{r}}J^\tau - NY^{\check{r}}}{J^\tau}\partial_{\check{r}}\alpha^i =$$

$$= N^{\check{r}}\partial_{\check{r}}\alpha^i + N\partial_{\check{r}}\alpha^i(T^{-1})^{\check{r}j}\Pi_j\frac{X}{\mu J^\tau}\Big[\frac{\partial f(x)}{\partial x}|_{x=\frac{X}{\sqrt{\gamma}}}\Big]^{-1},$$

$$\Downarrow$$

$$\rho_\mu(\tau,\vec{\sigma}) = -\frac{\partial L(\tau,\vec{\sigma})}{\partial z^\mu_\tau(\tau,\vec{\sigma})} = -\Big[l_\mu\frac{\partial L}{\partial N} - \epsilon z_{\check{s}\mu}{}^3g^{\check{s}\check{r}}\frac{\partial L}{\partial N^{\check{r}}}\Big](\tau,\vec{\sigma}) =$$

$$= \mu\sqrt{\gamma}\Big[l_\mu\frac{\partial N f(\frac{X}{\sqrt{\gamma}})}{\partial N} - \epsilon z_{\check{s}\mu}{}^3g^{\check{s}\check{r}}N\frac{\partial f(\frac{X}{\sqrt{\gamma}})}{\partial N^{\check{r}}}\Big](\tau,\vec{\sigma}) =$$

$$= \Big[\frac{\mu}{X}\Big(\sqrt{\gamma}Xf(\frac{X}{\sqrt{\gamma}}) + [(J^\tau)^2 - X^2]\frac{\partial f(x)}{\partial x}|_{x=\frac{X}{\sqrt{\gamma}}}\Big)l_\mu +$$

$$+ \frac{\mu}{X}\frac{\partial f(x)}{\partial x}|_{x=\frac{X}{\sqrt{\gamma}}}J^\tau Y^{\check{r}}z_{\check{r}\mu}\Big](\tau,\vec{\sigma}) =$$

$$= \Big[\frac{\mu}{X}\Big(\sqrt{\gamma}Xf(\frac{X}{\sqrt{\gamma}}) + [(J^\tau)^2 - X^2]\frac{\partial f(x)}{\partial x}|_{x=\frac{X}{\sqrt{\gamma}}}\Big)l_\mu\Big](\tau,\vec{\sigma}) -$$

$$- \epsilon\Big[J^\tau(T^{-1})^{\check{r}i}\Pi_i z_{\check{r}\mu}\Big](\tau,\vec{\sigma}) =$$

$$= \Big[\mu\sqrt{\gamma}G(X, J^\tau, \sqrt{\gamma})l_\mu + J^\tau(T^{-1})^{\check{r}i}\Pi_i z_{\check{r}\mu}\Big](\tau,\vec{\sigma}),$$

$$with$$

$$G(\frac{X}{\sqrt{\gamma}}, \frac{(J^\tau)^2}{\gamma}) = f(\frac{X}{\sqrt{\gamma}}) + \frac{\frac{(J^\tau)^2}{\gamma} - (\frac{X}{\sqrt{\gamma}})^2}{\frac{X}{\sqrt{\gamma}}}\frac{\partial f(x)}{\partial x}|_{x=\frac{X}{\sqrt{\gamma}}} = f(n) + \frac{\frac{(J^\tau)^2}{\gamma} - n^2}{n}\frac{\partial f(n)}{\partial n}. \tag{5.11}$$

To get the Hamiltonian expression of the constraints $\mathcal{H}^\mu(\tau,\vec{\sigma}) \approx 0$, we have to find the

solution X of the equation $X^2 + {}^3g_{\check{r}\check{s}}Y^{\check{r}}(X)Y^{\check{s}}(X) = (J^\tau)^2$ with $Y^{\check{r}}(X)$ given by the second line of Eq.(5.11). This equation may be written in the following forms

$$X^2\Big[\mu^2 + A^2\big(\frac{\partial f(x)}{\partial x}\big|_{x=\frac{X}{\sqrt{\gamma}}}\big)^{-2}\Big] = B^2, \quad or$$

$$\big(\frac{X}{\sqrt{\gamma}}\big)^2 \quad \Big[1 + \frac{A^2}{\mu^2 + A^2}\big((\frac{\partial f(x)}{\partial x}\big|_{x=\frac{X}{\sqrt{\gamma}}})^{-2} - 1\big)\Big] = \frac{B^2}{\gamma(\mu^2 + A^2)}, \quad or$$

$$n^2 \quad \Big[1 + \frac{A^2}{\mu^2 + A^2}\big((\frac{\partial f(n)}{\partial n})^{-2} - 1\big)\Big] = \frac{B^2}{\gamma(\mu^2 + A^2)},$$

$$A^2 = {}^3g_{\check{r}\check{s}}(T^{-1})^{\check{r}i}\Pi_i(T^{-1})^{\check{s}j}\Pi_j,$$
$$B^2 = \mu^2(J^\tau)^2,$$

$$\Rightarrow \quad X = \sqrt{\gamma}n = \sqrt{\gamma}F\big(\frac{A^2}{\mu^2}, \frac{(J^\tau)^2}{\gamma}\big) = \sqrt{\gamma}\tilde{F}\big(\frac{A^2}{\mu^2 + A^2}, \frac{B^2}{\gamma(\mu^2 + A^2)}\big),$$

$$\Rightarrow \quad \rho_\mu = \mu\sqrt{\gamma}\tilde{G}\big(\frac{A^2}{\mu^2 + A^2}, \frac{B^2}{\gamma(\mu^2 + A^2)}\big)l_\mu + J^\tau(T^{-1})^{\check{r}i}\Pi_i z_{\check{r}\mu} =$$
$$= \mathcal{M}l_\mu + \mathcal{M}^{\check{r}}z_{\check{r}\mu}. \tag{5.12}$$

Therefore, all the dependence on the metric and on the Lagrangian coordinates and their momenta is concentrated in the 3 functions $\sqrt{\gamma}$, $A^2/\mu^2 = {}^3g_{\check{r}\check{s}}(T^{-1})^{\check{r}i}\Pi_i(T^{-1})^{\check{s}j}\Pi_j/\mu^2$, $B^2/\mu^2\gamma = (J^\tau)^2/\gamma$.

Let us consider various cases.

1) $p = 0$, dust. As in Section II the equation for X and the constraints are

$$X^2 \quad [\mu^2 + A^2] = B^2,$$

$$X = \frac{B}{\sqrt{\mu^2 + A^2}} = \frac{\mu|J^\tau|}{\sqrt{\mu^2 + {}^3g_{\check{r}\check{s}}(T^{-1})^{\check{r}i}\Pi_i(T^{-1})^{\check{s}j}\Pi_j}}$$

$$\rightarrow_{A^2 \to 0} |J^\tau|[1 - \frac{A^2}{2\mu^2} + O(A^4)],$$

$$Y^{\check{r}} = -\frac{X}{\mu}(T^{-1})^{\check{r}i}\Pi_i = -\frac{|J^\tau|(T^{-1})^{\check{r}i}\Pi_i}{\sqrt{\mu^2 + {}^3g_{\check{r}\check{s}}(T^{-1})^{\check{r}i}\Pi_i(T^{-1})^{\check{s}j}\Pi_j}},$$

$$\Downarrow$$

$$\rho_\mu = |J^\tau|\sqrt{\mu^2 + {}^3g_{\check{r}\check{s}}(T^{-1})^{\check{r}i}\Pi_i(T^{-1})^{\check{s}j}\Pi_j}\,l_\mu + J^\tau(T^{-1})^{\check{r}i}\Pi_i z_{\check{r}\mu}. \tag{5.13}$$

2) $p = k\rho$, $k \neq -1$. The equation for X is

$$X^2 \quad \Big[\mu^2 + \frac{A^2}{(k+1)^2(\frac{X}{\sqrt{\gamma}})^{2k}}\Big] = B^2, \quad or$$

$$\big(\frac{X}{\sqrt{\gamma}}\big)^2 \quad \Big[1 + \frac{A^2}{\mu^2 + A^2}\big(\frac{1}{(k+1)^2(\frac{X}{\sqrt{\gamma}})^{2k}} - 1\big)\Big] = \frac{B^2}{\gamma(\mu^2 + A^2)},$$

$$Y^{\check{r}} = -\frac{\sqrt{\gamma}(\frac{X}{\sqrt{\gamma}})^{1-k}(T^{-1})^{\check{r}i}\Pi_i}{\mu(k+1)},$$

$$\rho_\mu = \mu\sqrt{\gamma}(\frac{X}{\sqrt{\gamma}})^{k-1}[(k+1)\frac{(J^\tau)^2}{\gamma} - k()\frac{X}{\sqrt{\gamma}}^2]l_\mu + J^\tau(T^{-1})^{\check{r}i}\Pi_i z_{\check{r}\mu}. \tag{5.14}$$

Let us define Z as the deviation of X from dust (for $A^2 \to 0$ $[\partial_\tau \alpha^i = 0]$: we have $Z \to 1$.

$X = \frac{|B|}{\sqrt{\mu^2+A^2}}Z$. Then we get the following equation for Z

$$Z^2\Big[\frac{\mu^2}{\mu^2+A^2} + \frac{A^2(\mu^2+A^2)^{k-1}\gamma^k}{(k+1)^2B^{2k}}Z^{-2k}\Big] = 1, \quad or$$
$$Z^2[\alpha^2 + \beta_k^2 Z^{-2k}] = 1,$$

$$\alpha^2 = \frac{\mu^2}{\mu^2+A^2} \to_{A^2 \to 0} 1,$$
$$\beta_k^2 = \frac{A^2(\mu^2+A^2)^{k-1}\gamma^k}{(k+1)^2B^{2k}} \to_{A^2 \to 0} 0. \tag{5.15}$$

We may consider the following subcases:
2a) $k = m \neq -1$, with the equation

$$Z_1 = Z^2, \qquad X = \frac{|B|}{\sqrt{\mu^2+A^2}}\sqrt{Z_1},$$

$$Z_1 \quad [\alpha^2 + \beta_m^2 Z_1^{-m}] = 1, \quad or \quad \beta_m^2 Z_1^{1-m} + \alpha^2 Z_1 - 1 = 0. \tag{5.16}$$

i) $p = \rho$ $(k = m = 1)$, with the equation

$$Z_1 = \frac{1-\beta_1^2}{\alpha^2} = \frac{\gamma(\mu^2+A^2)(4\frac{B^2}{\gamma}-A^2)}{4\mu^2 B^2},$$

$$X = \frac{\sqrt{4\frac{B^2}{\gamma}-A^2}}{2\mu} = \frac{1}{2\mu}\sqrt{4\mu^2\frac{(J^\tau)^2}{\gamma} + {}^3g_{\check{r}\check{s}}(T^{-1})^{\check{r}i}\Pi_i(T^{-1})^{\check{s}j}\Pi_j}, \quad for \quad A^2 < 4\frac{B^2}{\gamma}. \tag{5.17}$$

ii) $p = 2\rho$ $(k = m = 2)$, with the equation

$$\alpha^2 Z_1^2 - Z_1 + \beta_2^2 = 0, \qquad Z_1 = \frac{1}{2\alpha^2}[1 \pm \sqrt{1-4\alpha^2\beta_2^2}], \tag{5.18}$$

iii) $p = -2\rho$ $(k = m = -2)$, with the equation

$$\beta_{-2}^2 Z_1^3 + \alpha^2 Z_1 - 1 = 0, \tag{5.19}$$

2b) $k = \frac{1}{m}$, with the equation

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$$Z_2 = Z^{\frac{2}{m}}, \qquad X = \frac{|B|}{\sqrt{\mu^2 + A^2}} Z_2^{\frac{m}{2}},$$

$$\alpha^2 Z_2^m + \beta_{1/m}^2 Z_2^{m-1} - 1 = 0, \tag{5.20}$$

i) $p = \frac{1}{2}\rho$ ($k = \frac{1}{2}$, $m = 2$), with the equation

$$\alpha^2 Z_2^2 + \beta_{1/2}^2 Z_2 - 1 = 0,$$

$$Z_2 = \frac{1}{2\alpha^2}[-\beta_{1/2}^2 \pm \sqrt{\beta_{1/2}^4 + 4\alpha^2}]. \tag{5.21}$$

ii) $p = -\frac{1}{2}\rho$ ($k = -\frac{1}{2}$, $m = -2$), with the equation

$$\beta_{-1/2}^2 (Z_2^{-1})^3 + \alpha^2 (Z_2^{-1})^2 - 1 = 0. \tag{5.22}$$

iii) $p = \frac{1}{3}\rho$, photon gas ($k = \frac{1}{3}$, $m = 3$), $\beta_{1/3}^2 = \frac{9A^2 \gamma^{1/3}}{16 B^{2/3}(\mu^2 + A^2)^{2/3}}$, with the equation

$$\alpha^2 Z_2^3 + \beta_{1/3}^2 Z_2^2 - 1 = 0, \qquad or \quad Z_2^3 + p Z_2^2 + r = 0,$$

$$with \quad p = \frac{\beta_{1/3}^2}{\alpha^2} > 0, \quad r = -\frac{1}{\alpha^2} < 0,$$

$$Z_2 = Y_2 - \frac{1}{3}p = Y_2 - \frac{\beta_{1/3}^2}{3\alpha^2} \to_{A^2 \to 0} 1,$$

$$X = \frac{|B|}{\sqrt{\mu^2 + A^2}}(Y_2 - \frac{\beta_{1/3}^2}{3\alpha^2}) = \frac{|B|}{\sqrt{\mu^2 + A^2}}(Y_2 - \frac{3A^2(\mu^2 + A^2)^{1/3}\gamma^{1/3}}{16 B^{2/3}})^{3/2},$$

$$Y_2^3 + a Y_2 + b = 0,$$

$$a = -\frac{1}{3}p^2 = -\frac{\beta_{1/3}^4}{3\alpha^4} = -\frac{27 A^4 (\mu^2 + A^2)^{2/3}\gamma^{2/3}}{2^8 B^{4/3}} < 0,$$

$$b = \frac{1}{27}(2p^3 + 27r) = \frac{1}{27\alpha^2}(2\frac{\beta_{1/3}^6}{\alpha^4} - 27) = \frac{\mu^2 + A^2}{\mu^2}\left(\frac{27 A^6 \gamma}{2^{11}\mu^4 B^2} - 1\right),$$

$$\frac{b^2}{4} + \frac{a^3}{27} = \frac{1}{4 \cdot 27\alpha^4}(27 - 4\frac{\beta_{1/3}^6}{\alpha^4}) = \frac{(\mu^2 + A^2)^2}{4\mu^4}\left(1 - \frac{27 A^6 \gamma}{2^{10}\mu^4 B^2}\right) \to_{A^2 \to 0} \frac{1}{4} > 0,$$

$$Y_2 = \left(-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right)^{1/3} - \left(\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right)^{1/3} \to_{A^2 \to 0} 1,$$

$$\Downarrow$$

$$X = |J^\tau|\Big[-\frac{3 A^2 \gamma^{1/3}}{16 (J^\tau)^{2/3}} + \frac{1}{2^{1/3}}\Big(1 - \frac{27 A^6 \gamma}{2^{11}\mu^6 (J^\tau)^2} + \sqrt{1 - \frac{27 A^6 \gamma}{2^{10}\mu^6 (J^\tau)^2}}\Big)^{1/3} -$$

$$- \frac{1}{2^{1/3}}\Big(-1 + \frac{27 A^6 \gamma}{2^{11}\mu^6 (J^\tau)^2} + \sqrt{1 - \frac{27 A^6 \gamma}{2^{10}\mu^6 (J^\tau)^2}}\Big)^{1/3}\Big]^{3/2} \to_{A^2 \to 0}$$

$$\to_{A^2 \to 0} |J^\tau|[1 - \frac{9 A^2 \gamma^{1/3}}{32 (J^\tau)^{2/3}} + O(A^4)],$$

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$$\Downarrow$$

$$\begin{aligned}
\rho_\mu &= \frac{1}{3}\mu\sqrt{\gamma}\Big(\frac{X}{\sqrt{\gamma}}\Big)^{-2/3}\Big[4\frac{(J^\tau)^2}{\gamma} - \Big(\frac{X}{\sqrt{\gamma}}\Big)^2\Big]l_\mu + J^\tau(T^{-1})^{\check{r}i}\Pi_i z_{\check{r}\mu} = \\
&= \mathcal{M}l_\mu + \mathcal{M}^r z_{r\mu}.
\end{aligned} \tag{5.23}$$

In the case of the photon gas we get a closed analytical form for the constraints.

3) $p = k\rho^\gamma$, $\gamma = 1 + \frac{1}{n}$, $\gamma \neq 1$ $(n \neq 0)$, with the equation

$$X^2 \quad \Big[\mu^2 + A^2\Big(1 - k(\mu\frac{X}{\sqrt{\gamma}})^{\gamma-1}\Big)^{\frac{2\gamma}{\gamma-1}}\Big] = B^2, \quad or$$

$$X^2 \quad \Big[\mu^2 + A^2\Big(1 - k(\mu\frac{X}{\sqrt{\gamma}})^{\frac{1}{n}}\Big)^{2(n+1)}\Big] = B^2, \quad or$$

$$\Big(\frac{X}{\sqrt{\gamma}}\Big)^2 \quad \Big[1 + \frac{A^2}{\mu^2 + A^2}\Big([1 - k(\mu\frac{X}{\sqrt{\gamma}})^{\frac{1}{n}}]^{2(n+1)} - 1\Big)\Big] = \frac{B^2}{\gamma(\mu^2 + A^2)},$$

$$Y^{\check{r}} = -\frac{X}{\mu}[1 - k(\frac{X}{\sqrt{\gamma}})^{\frac{1}{n}}]^{n+1}(T^{-1})^{\check{r}i}\Pi_i,$$

$$\rho_\mu = \mu\sqrt{\gamma}\frac{\frac{(J^\tau)^2}{\gamma} - k\mu^{\frac{1}{n}}(\frac{X}{\sqrt{\gamma}})^{2+\frac{1}{n}}}{\frac{X}{\sqrt{\gamma}}[1 - k(\mu\frac{X}{\sqrt{\gamma}})^{\frac{1}{n}}]^{n+1}}l_\mu + J^\tau(T^{-1})^{\check{r}i}\Pi_i z_{\check{r}\mu}. \tag{5.24}$$

Let us define Z as the deviation of X from dust (for $A^2 \to 0$ $[\partial_\tau \alpha^i = 0]$: we have $Z \to 1$, $X = \frac{|B|}{\sqrt{\mu^2 + A^2}}Z$). Then we get the following equation for Z

$$Z^2\Big[\frac{\mu^2}{\mu^2 + A^2} + \frac{A^2(\mu^2 + A^2)^{k-1}\gamma^k}{(k+1)^2 B^{2k}}\Big(1 - k(\frac{\mu|B|Z}{\sqrt{\gamma}\sqrt{\mu^2 + A^2}})^{\frac{1}{n}}\Big)^{2(n+1)}\Big] = 1. \tag{5.25}$$

In conclusion, only in the cases of the dust and of the photon gas we get the closed analytic form of the constraints (i.e. of the density of invariant mass $\mathcal{M}(\tau, \vec{\sigma})$, because for the momentum density we have $\mathcal{M}^{\check{r}} = J^\tau(T^{-1})^{\check{r}i}\Pi_i$ independently from the type of perfect fluid).

In all the other cases we have only an implicit form for them depending on the solution X of Eq.(5.12) and numerical methods should be used.

VI. COUPLING TO ADM METRIC AND TETRAD GRAVITY.

Let us now assume to have a globally hyperbolic, asymptotically flat at spatial infinity spacetime M^4 with the spacelike leaves Σ_τ of the foliations associated with its 3+1 splittings, diffeomorphic to R^3 [25–27,11].

In σ_τ-adapted coordinates $\sigma^A = (\sigma^\tau = \tau; \vec{\sigma})$ corresponding to a holonomic basis $[d\sigma^A, \partial_A = \partial/\partial\sigma^A]$ for tensor fields we have [25] [N and N^r are the lapse and shift functions; $^3g_{rs}$ is the 3-metric of Σ_τ; $l^A(\tau, \vec{\sigma})$ is the unit normal vector fields to Σ_τ]

$$^4g_{AB} = g_{AB} = \{^4g_{\tau\tau} = \epsilon(N^2 - {}^3g_{rs}N^rN^s); {}^4g_{\tau r} = -\epsilon\,{}^3g_{rs}N^s; {}^4g_{rs} = -\epsilon\,{}^3g_{rs}\} =$$
$$= \epsilon l_A l_B + \triangle_{AB},$$

$$\triangle_{AB} = {}^4g_{AB} - \epsilon l_A l_B. \tag{6.1}$$

A set of Σ_τ-adapted tetrad and cotetrad fields is $(a) = (1), (2), (3); {}^3e^r_{(a)}$ and $^3e^{(a)}_r = {}^3e_{(a)r}$ are triad and cotriad fields on Σ_τ]

$$^4_{(\Sigma)}E^A_{(o)} = \epsilon l^A = (\frac{1}{N}; -\frac{N^r}{N}),$$
$$^4_{(\Sigma)}E^A_{(a)} = (0; {}^3e^r_{(a)}),$$

$$^4_{(\Sigma)}E^{(o)}_A = l_A = (N; \vec{0}),$$
$$^4_{(\Sigma)}E^{(a)}_A = (N^{(a)} = N^r\,{}^3e^{(a)}_r; {}^3e^{(a)}_r). \tag{6.2}$$

In these coordinates the energy-momentum tensor of the perfect fluid is

$$T^{AB} = -\epsilon(\rho + p)U^AU^B + p\,{}^4g^{AB}. \tag{6.3}$$

If we use the notation $\Gamma = \epsilon l_A U^A = \epsilon N U^\tau$, we get

$$T_{AB} = {}^4g_{AC}\,{}^4g_{BC}T^{CD} = [\epsilon l_A l_C + \triangle_{AC}][\epsilon l_B l_D + \triangle_{BD}]T^{CD} =$$
$$= E l_A l_B + j_A l_B + l_A j_B + \mathcal{S}_{AB},$$

$$E = T^{AB}l_A l_B = -\epsilon[(\rho + p)\Gamma^2 - p],$$
$$j_A = \epsilon\triangle_{AC}T^{CB}l_B = -\epsilon(\rho + p)\Gamma\triangle_{AB}U^B,$$
$$\mathcal{S}_{AB} = \triangle_{AC}\triangle_{BD}T^{CD} = -\epsilon[(\rho + p)\triangle_{AC}U^C\triangle_{BD}U^D - p\triangle_{AB}]. \tag{6.4}$$

The same decomposition can be referred to a non-holonomic basis $[\theta^{\bar{A}} = \{\theta^l = Nd\tau; \theta^r = d\sigma^r + N^r d\tau\}, X_{\bar{A}} = \{X_l = N^{-1}(\partial_\tau - N^r\partial_r); \partial_r\}; \bar{A} = (l; r)]$ in which we have $[\bar{\Gamma} = \epsilon \bar{l}_{\bar{A}}\bar{U}^{\bar{A}} = \epsilon\bar{U}^l = \sqrt{1 - {}^3g_{rs}\bar{U}^r\bar{U}^s}$, so that $\bar{U}^r$ may be interpreted as the generalized boost velocity of $\bar{U}^{\bar{A}}$ with respect to $\bar{l}^{\bar{A}}$ [28,29]]

$$^4\bar{g}_{\bar{A}\bar{B}} = \{^4\bar{g}_{ll} = \epsilon; {}^4\bar{g}_{lr} = 0; {}^4\bar{g}_{rs} = {}^4g_{rs} = -\epsilon\,{}^3g_{rs}\} =$$
$$= \epsilon\bar{l}_{\bar{A}}\bar{l}_{\bar{B}} + \bar{\triangle}_{\bar{A}\bar{B}},$$

$$\bar{l}^{\bar{A}} = (\epsilon; \vec{0}), \qquad \bar{l}_{\bar{A}} = (1; \vec{0}),$$

$$\bar{\triangle}_{ll} = \bar{\triangle}_{lr} = 0, \qquad \bar{\triangle}_{rs} = -\epsilon \, {}^{3}g_{rs}, \qquad \bar{\triangle}_{\bar{A}\bar{B}} \bar{U}^{\bar{B}} = (0; -\epsilon \, {}^{3}g_{rs}),$$

$$\bar{T}_{\bar{A}\bar{B}} = -\epsilon(\rho + p)\bar{U}_{\bar{A}}\bar{U}_{\bar{B}} + p \, {}^{4}\bar{g}_{\bar{A}\bar{B}} =$$
$$= \bar{E}\bar{l}_{\bar{A}}\bar{l}_{\bar{B}} + \bar{j}_{\bar{A}}\bar{l}_{\bar{B}} + \bar{l}_{\bar{A}}\bar{j}_{\bar{B}} + \bar{\mathcal{S}}_{\bar{A}\bar{B}},$$

$$\bar{E} = \bar{T}^{\bar{A}\bar{B}}\bar{l}_{\bar{A}}\bar{l}_{\bar{B}} = -\epsilon[(\rho + p)\bar{\Gamma}^2 - p],$$
$$\bar{j}_{\bar{A}} = \epsilon\bar{\triangle}_{\bar{A}\bar{C}}\bar{T}^{\bar{C}\bar{B}}\bar{l}_{\bar{B}} = (\bar{j}_l = 0; \bar{j}_r = -\epsilon(\rho + p)\bar{\Gamma} \, {}^{3}g_{rs}\bar{U}^s),$$
$$\bar{\mathcal{S}}_{\bar{A}\bar{B}} = \bar{\triangle}_{\bar{A}\bar{C}}\bar{\triangle}_{\bar{B}\bar{D}}\bar{T}^{\bar{C}\bar{D}} =$$
$$= (\bar{\mathcal{S}}_{ll} = \bar{\mathcal{S}}_{lr} = 0; \bar{\mathcal{S}}_{rs} = -\epsilon[-p \, {}^{3}g_{rs} + (\rho + p){}^{3}g_{ru}\bar{U}^u \, {}^{3}g_{sv}\bar{U}^v)]. \tag{6.5}$$

$\bar{E}$ and $\bar{j}_r$ are the energy and momentum densities determined by the Eulerian observers on Σ_τ, while $\bar{\mathcal{S}}_{rs}$ is called the "spatial stress tensor".

The non-holonomic basis is used to get the 3+1 decomposition (projection normal and parallel to Σ_τ) of Einstein's equations with matter ${}^{4}G^{\mu\nu} \overset{\circ}{=} \frac{\epsilon c^3}{8\pi G}T^{\mu\nu}$ [when one does not has an action principle for matter, one cannot use the Hamiltonian ADM formalism]: in this way one gets four restrictions on the Cauchy data [${}^{4}\bar{G}^{ll} \overset{\circ}{=} \frac{\epsilon c^3}{8\pi G}\bar{T}^{ll}$ and ${}^{4}\bar{G}^{lr} \overset{\circ}{=} \frac{\epsilon c^3}{8\pi G}\bar{T}^{lr}$; they become the secondary first class superhamiltonian and supermomentum constraints in the ADM theory; $k = c^3/8\pi G$]

$$\left[{}^{3}R + ({}^{3}K)^2 - {}^{3}K_{rs}\,{}^{3}K^{rs}\right](\tau, \vec{\sigma}) \overset{\circ}{=} \frac{2}{k}\bar{E}(\tau, \vec{\sigma}),$$

$$({}^{3}K^{rs} - {}^{3}g^{rs}\,{}^{3}K)_{|s}(\tau, \vec{\sigma}) \overset{\circ}{=} \frac{1}{k}\bar{J}^r(\tau, \vec{\sigma}), \tag{6.6}$$

and the spatial Einstein's equations ${}^{4}\bar{G}^{rs} \overset{\circ}{=} \frac{\epsilon c^3}{8\pi G}\bar{T}^{rs}$. By introducing the extrinsic curvature ${}^{3}K_{rs}$, this last equations are written in a first order form [it corresponds to the Hamilton equations of the ADM theory for ${}^{3}g_{rs}$ and ${}^{3}\tilde{\Pi}^{rs} = \frac{\epsilon c^3}{8\pi G}\sqrt{\gamma}({}^{3}K^{rs} - {}^{3}g^{rs}\,{}^{3}K)$; "|" denotes the covariant 3-derivative]

$$\partial_\tau\,{}^{3}g_{rs}(\tau, \vec{\sigma}) = \left[N_{r|s} + N_{s|r} - 2N\,{}^{3}K_{rs}\right](\tau, \vec{\sigma}),$$

$$\partial_\tau\,{}^{3}K_{rs}(\tau, \vec{\sigma}) \overset{\circ}{=} \left(N[{}^{3}R_{rs} + {}^{3}K\,{}^{3}K_{rs} - 2\,{}^{3}K_{ru}\,{}^{3}K^u{}_s] - \right.$$
$$\left. - N_{|s|r} + N^u{}_{|s}\,{}^{3}K_{ur} + N^u{}_{|r}\,{}^{3}K_{us} + N^u\,{}^{3}K_{rs|u}\right)(\tau, \vec{\sigma}) - $$
$$- \frac{1}{k}\left[(\bar{\mathcal{S}}_{rs} + \frac{1}{2}{}^{3}g_{rs}(\bar{E} - \bar{\mathcal{S}}^u{}_u)\right](\tau, \vec{\sigma}). \tag{6.7}$$

The matter equations $T^{\mu\nu}{}_{;\nu} \overset{\circ}{=} 0$ become a generalized continuity equation [entropy conservation when the particle number conservation law is added to the system] $\bar{l}_{\bar{A}}T^{\bar{A}\bar{B}}{}_{;\bar{B}} \overset{\circ}{=} 0$ and generalized Euler equations $\bar{\triangle}_{\bar{A}\bar{B}}T^{\bar{B}\bar{C}}{}_{;\bar{C}} \overset{\circ}{=} 0$ [$\mathcal{L}_X$ is the Lie derivative with respect to the vector field X]

$$\left[\partial_\tau\bar{E} + N\bar{j}^r{}_{|r}\right] \overset{\circ}{=} \left[N(\bar{\mathcal{S}}^{rs}\,{}^{3}K_{rs} + \bar{E}\,{}^{3}K) - 2\bar{j}^r N_{|r} + \mathcal{L}_{\vec{N}}\bar{E}\right](\tau, \vec{\sigma}),$$

$$\left[\partial_\tau\bar{j}_r + N\bar{\mathcal{S}}^{rs}{}_{|s}\right](\tau, \vec{\sigma}) \overset{\circ}{=} \left[N(2\,{}^{3}K^{rs}\bar{j}_s + \bar{j}^r\,{}^{3}K) - \bar{\mathcal{S}}^{rs}N_{|s} - \bar{E}N^{|r} + \mathcal{L}_{\vec{N}}\bar{j}^r\right](\tau, \vec{\sigma}). \tag{6.8}$$

The equation for $\bar{\mathcal{S}}_{rs}$ would follow from an equation of state or dynamical equation of the sources [for perfect fluids it is the particle number conservation].

This formulation is the starting point of many approaches to the post-Newtonian approximation (see for instance Refs. [23,30]) and to numerical gravity (see for instance Refs. [31,32]).

Instead with the action principle for the perfect fluid described with Lagrangian coordinates (containing the information on the equation of state and on the particle number and entropy conservations) coupled to the action for tetrad gravity of Ref. [25] (it is the ADM action of metric gravity re-expressed in terms of a new parametrization of tetrad fields) we get

$$S = -\epsilon k \int d\tau d^3\sigma \, \{ N \, {}^3e \, \epsilon_{(a)(b)(c)} \, {}^3e^r_{(a)} \, {}^3e^s_{(b)} \, {}^3\Omega_{rs(c)} +$$

$$+ \frac{{}^3e}{2N} ({}^3G_o^{-1})_{(a)(b)(c)(d)} \, {}^3e^r_{(b)} (N_{(a)|r} - \partial_\tau \, {}^3e_{(a)r}) \, {}^3e^s_{(d)} (N_{(c)|s} - \partial_\tau \, {}^3e_{(c) \ s}) \} (\tau, \vec{\sigma}) +$$

$$- \int d\tau d^3\sigma \{ N\sqrt{\gamma}\rho(\frac{|J|}{N\sqrt{\gamma}}, s) \}(\tau, \vec{\sigma}). \tag{6.9}$$

The superhamiltonian and supermomentum constraints of Ref. [25] are modified in the following way by the presence of the perfect fluid

$$\mathcal{H} = \mathcal{H}_o + \mathcal{M} \approx 0,$$
$$\mathcal{H}_r \approx \Theta_r = \Theta_{or} + \mathcal{M}_r \approx 0, \tag{6.10}$$

with $\mathcal{M}_r = J^\tau (T^{-1})^{ri}\Pi_i = -\partial_r \alpha^i \Pi_i$ and with $\mathcal{M}$ given by Eq.(2.12) for the dust and by Eq.(5.23) for the photon gas.

In the case of dust the explicit Hamiltonian form of the energy and momentum densities is

$$\mathcal{M}(\tau, \vec{\sigma}) = J^\tau(\tau, \vec{\sigma})\sqrt{\mu^2 + [{}^3g_{rs}(T^{-1}(\alpha))^{ri}(T^{-1}(\alpha))^{sj}\Pi_i\Pi_j](\tau, \vec{\sigma})} =$$

$$= -det\,(\partial_r \alpha^i(\tau, \vec{\sigma}))\sqrt{\mu^2 + {}^3g^{uv}\frac{\partial_u\alpha^m\partial_v\alpha^n}{[det\,(\partial_r\alpha^k)]^2}\Pi_m\Pi_n\Big](\tau, \vec{\sigma})},$$

$$\mathcal{M}_r(\tau, \vec{\sigma}) = {}^3g_{rs}J^\tau(\tau, \vec{\sigma})[(T^{-1}(\alpha))^{si}\Pi_i](\tau, \vec{\sigma}) = -\partial_r\alpha^i(\tau, \vec{\sigma})\Pi_i(\tau, \vec{\sigma}). \tag{6.11}$$

In any case, all the dependence of $\mathcal{M}$ and $\mathcal{M}^r$ on the metric and on the Lagrangian coordinates and their momenta is concentrated in the 3 functions $\sqrt{\gamma}$, $A^2/\mu^2 = {}^3g_{\check{r}\check{s}}(T^{-1})^{\check{r}i}\Pi_i(T^{-1})^{\check{s}j}\Pi_j/\mu^2 = \frac{{}^3g^{uv}\partial_u\alpha^i\partial_v\alpha^j\Pi_i\Pi_J-J}{\mu^2[det\,(\partial_u\alpha^k)]^2}$, $B^2/\mu^2\gamma = (J^\tau)^2/\gamma = \frac{1}{\gamma}[det\,(\partial_r\alpha^i)]^2$.

The study of the canonical reduction to the 3-orthogonal gauges [26,27] will be done in a future paper.

VII. NON-DISSIPATIVE ELASTIC MATERIALS.

With the same formalism we may describe relativistic continuum mechanics [any relativistic material (non-homogeneous, pre-stressed,...) in the non-dissipative regime] and in particular a relativistic elastic continuum [6] [see also Refs. [33,34] and their bibliography] in the rest-frame instant form of dynamics.

Now the scalar fields $\tilde{\alpha}^i(z(\tau,\vec{\sigma})) = \alpha^i(\tau,\vec{\sigma})$ describe the idealized "molecules" of the material in an abstract 3-dimensional manifold called the "material space", while $J^{\check{A}}(\tau,\vec{\sigma}) = [N\sqrt{\gamma}nU^{\check{A}}](\tau,\vec{\sigma})$ is the matter number current with future-oriented timelike 4-velocity vector field $U^{\check{A}}(\tau,\vec{\sigma})$; n is a scalar field describing the local rest-frame matter number density. The quantity $\partial_{\check{A}}\alpha^i(\tau,\vec{\sigma}) = z^\mu_{\check{A}}(\tau,\vec{\sigma})\partial_\mu\tilde{\alpha}^i(z)$ is called the "relativistic deformation gradient" in Σ_τ-adapted coordinates.

The material space inherits a Riemannian (symmetric and positive definite) 3-metric from the spacetime M^4

$$G^{ij} = {}^4g^{\mu\nu}\partial_\mu\tilde{\alpha}^i\partial_\nu\tilde{\alpha}^j = {}^4g^{\check{A}\check{B}}\partial_{\check{A}}\alpha^i\partial_{\check{B}}\alpha^j. \tag{7.1}$$

Its inverse G_{ij} carries the information about the actual distances of adjacent molecules in the local rest frame.

For an ideal fluid the 3-form η of Eq(1.3) gives the volume element in the material space, which is sufficient to describe the mechanical properties of an ideal fluid. Since we have that $n = \frac{|J|}{N\sqrt{\gamma}}$ is a scalar, we can evaluate n in the local rest frame at z where $U^\mu(z) = (\frac{1}{\sqrt{\epsilon\,{}^4\bar{g}_{\bar{o}\bar{o}}}};\vec{0})$ and $\partial_o\tilde{\alpha}^i|_z = 0$ [in the local adapted non-holonomic basis we have ${}^4\bar{g}_{\check{A}\check{B}} = \epsilon\begin{pmatrix} 1 & 0 \\ 0 & -{}^3g_{rs} \end{pmatrix}$, ${}^4\bar{g}^{\bar{A}\bar{B}} = \epsilon\begin{pmatrix} 1 & 0 \\ 0 & -{}^3g^{rs} \end{pmatrix}$, $\sqrt{{}^4\bar{g}} = \sqrt{\gamma}/\sqrt{\epsilon\,{}^4\bar{g}^{\bar{o}\bar{o}}}$, $\sqrt{{}^4\bar{g}^{-1}} = \sqrt{{}^3\bar{g}^{-1}}/\sqrt{\epsilon\,{}^4\bar{g}_{\bar{o}\bar{o}}} = \sqrt{\epsilon\,{}^4\bar{g}_{\bar{o}\bar{o}}}/\sqrt{{}^4\bar{g}} = \sqrt{{}^4\bar{g}^{-1}}$]. We get at z

$$n = \frac{J^o}{U^o\sqrt{{}^4\bar{g}}} = \eta_{123}\,det\,\partial_k\tilde{\alpha}^i\,\frac{\sqrt{\epsilon\,{}^4\bar{g}_{\bar{o}\bar{o}}}}{\sqrt{{}^4\bar{g}}} = \eta_{123}\,det\,\partial_k\tilde{\alpha}^i\,\sqrt{det\,|{}^4\bar{g}^{rs}|} = \eta_{123}\sqrt{det\,G^{ij}}. \tag{7.2}$$

Therefore, we have $n = \eta_{123}\sqrt{det\,G^{ij}}$.

Moreover, the material space of elastic materials, which have not only volume rigidity but also shape rigidity, is equipped with a Riemannian (symmetric and definite positive) 3-metric $\gamma^{(M)}_{ij}(\alpha^i)$, the "material metric", which is frozen in the material and it is not a dynamical object of the theory. It describes the "would be" local rest-frame space distance between neighbouring "molecules", measured in the locally relaxed state of the material. To measure the components $\gamma^{(M)}_{ij}(\alpha^i)$ we have to relax the material at different points $\alpha^i(\tau,\vec{\sigma})$ separately, since global relaxation of the material may not be possible [the material space may not be isometric with any 3-dimensional subspace of M^4, as in classical non-linear elastomechanics, when the material exhibits internal stresses frozen in it]. The components $\gamma^{(M)}_{ij} = \gamma^{(M)}_{ij}(\alpha^i(\tau,\vec{\sigma}))$ are given functions, which describe axiomatically the properties of the material (the theory is fully invariant with respect to reparametrizations of the material space).

For ideal fluids $\gamma^{(M)}_{ij} = \delta_{ij}$; this also holds for non-pre-stressed materials without "internal" or "frozen" stresses.

Now the material space volume element η has

$$\eta_{123}(\alpha^i) = \sqrt{det\, \gamma_{ij}^{(M)}(\alpha^i)}, \qquad so\ that \quad n = \eta_{123}\sqrt{det\, G^{ij}} = \sqrt{det\, \gamma_{ij}^{(M)}}\sqrt{det\, G^{ij}}, \qquad (7.3)$$

and it cannot be put $= 1$ like for perfect fluids.

The pull-back of the material metric $\gamma_{ij}^{(M)}$ to M^4 is

$$\gamma_{\check{A}\check{B}}^{(M)} = \gamma_{ij}^{(M)}\partial_{\check{A}}\alpha^i\partial_{\check{B}}\alpha^j \qquad satisfying \quad \gamma_{\check{A}\check{B}}^{(M)}U^{\check{B}} = 0. \qquad (7.4)$$

The next step is to define a measure of the difference between the induced 3-metric $G_{ij}(\partial\alpha^i)$ and the constitutive metric $\gamma_{ij}^{(M)}(\alpha^i)$, to be taken as a measure of the deformation of the material and as a definition of a "relativistic strain tensor", locally vanishing when there is a local relaxation of the material. Some existing proposal for such a tensor in M^4 are [33,34]:

$$i) \qquad S_{\check{A}\check{B}}^{(1)} = \frac{1}{2}(^4g_{\check{A}\check{B}} - \epsilon U_{\check{A}}U_{\check{B}} - \gamma_{\check{A}\check{B}}^{(M)}) \qquad (7.5)$$

which vanishes at relax and satisfies $S_{\check{A}\check{B}}^{(1)}U^{\check{B}} = 0$ [but it must satisfy the involved matrix inequality $2\,det\,S^{(1)} \geq det\,(^4g - \epsilon UU)$].

$$ii) \qquad S^{(2)\check{A}}{}_{\check{B}} = \frac{1}{2}(K^{\check{A}}{}_{\check{B}} - \delta_{\check{B}}^{\check{A}}), \quad S^{(2)\check{A}}{}_{\check{B}}U^{\check{B}} = 0, \qquad (7.6)$$

with $K^{\check{A}}{}_{\check{B}} = {}^4g^{\check{A}\check{C}}(\gamma_{\check{C}\check{B}}^{(M)} - \epsilon U_{\check{C}}U_{\check{B}})$, $K^{\check{A}}{}_{\check{B}}U^{\check{B}} = U^{\check{A}}$ [the 4-velocity field is an eigenvector of the K-matrix].

$$iii) \qquad S^{(3)} = -\frac{1}{2}ln\,K, \qquad (7.7)$$

with the same K-matrix as in ii).

However a simpler proposal [6] is to define a "relativistic strain tensor" in the material space

$$S_i{}^j = \gamma_{ik}^{(M)}G^{kj}, \qquad (7.8)$$

with locally $S_i{}^j = \delta_i^j$ when there is local relaxation of the material (in this case physical spacelike distances between material points near a point $z^\mu(\tau,\vec{\sigma})$ agree with their material distances). Since $n = \eta_{123}\sqrt{det\, G^{ij}} = \sqrt{det\, \gamma_{ij}^{(M)}}\sqrt{det\, G^{ij}}$, we have $n = \sqrt{det\, S_i{}^j}$ for the local rest-frame matter number density.

We can now define the local rest-frame energy per unit volume of the material $n(\tau,\vec{\sigma})e(\tau,\vec{\sigma})$, where e denotes the molar local rest-frame energy (moles = number of particles)

$$e(\tau,\vec{\sigma}) = m + u_I(\tau,\vec{\sigma}). \qquad (7.9)$$

Here m is the molar local rest mass, u_I is the amount of internal energy (per mole of the material) of the elastic deformations, accumulated in an infinitesimal portion during the deformation from the locally relaxed state to the actual state of strain.

For isotropic media u_I may depend on the deformation only via the invariants of the strain tensor.

Let us notice that for an anisotropic material (like a crystal) the energy u_I may depend upon the orientation of the deformation with respect to a specific axis, reflecting the microscopic composition of the material: this information may be encoded in a vector field $E^i(\tau, \vec{\sigma})$ in the material space and one may assume $u_I = u_I(G_{ij}^{-1} E^i E^j)$.

The function $e = e[\alpha^i, G^{ij}, \gamma_{ij}^{(M)}, ...]$ describes the dependence of the energy of the material upon its state of strain and plays the role of an "equation of state" or "constitutive equation" of the material.

In the weak strain approximation of an isotropic elastic continuum (Hooke approximation) the function u_I depends only on the linear ($h = S_i{}^i$) and quadratic ($q = S_i{}^j S^j{}_i$) invariants of the strain tensor and coincides with the standard formula of linear elasticity [$V = \frac{1}{n} = \frac{1}{\sqrt{\det S_i{}^j}}$ is the specific volume],

$$u_I = \lambda(V)h^2 + 2\mu(V)q + O(cubic\,invariants), \tag{7.10}$$

where λ and μ are the Lamé coefficients.

The action principle for this description of relativistic materials is

$$S[^4g, \alpha^i, \partial\alpha^i] = \int d\tau d^3\sigma L(\tau, \vec{\sigma}) = -\int d\tau d^3\sigma (N\sqrt{\gamma})(\tau, \vec{\sigma})\, n(\tau, \vec{\sigma})\, e(\tau, \vec{\sigma}). \tag{7.11}$$

It is shown in Refs. [6,33] that the canonical stress-energy- momentum tensor $T_{\check{B}}^{\check{A}} = p_i^{\check{A}} \partial_{\check{B}}\alpha^i - \delta_{\check{B}}^{\check{A}} L$, where $p_i^{\check{A}} = -\frac{\partial L}{\partial \partial_{\check{A}}\alpha^i}$ is the relativistic Piola-Kirchhoff momentum density, and coincides with the symmetric energy-momentum tensor $T_{\check{A}\check{B}} = -2\frac{\partial L}{\partial \,^4g^{\check{A}\check{B}}}$, which satisfies $T^{\check{A}\check{B}}{}_{;\check{B}} = 0$ due to the Euler-Lagrange equations. This energy-momentum tensor may be written in the following form

$$T_{\check{A}\check{B}} = N\sqrt{\gamma}\, n\, [eU_{\check{A}\check{B}} + Z_{\check{A}\check{B}}], \tag{7.12}$$

where $Z_{\check{A}\check{B}} = Z_{ij}\, \partial_{\check{A}}\alpha^i \partial_{\check{B}}\alpha^j$ is the pull-back from the material space to M^4 of the "response tensor" of the material

$$Z_{ij} = 2\frac{\partial e}{\partial G^{ij}}, \qquad so\,that \quad de(G) = \frac{1}{2}Z_{ij}dG^{ij}. \tag{7.13}$$

The part $\tau_{\check{A}\check{B}} = nZ_{\check{A}\check{B}}$, $\tau_{\check{A}\check{B}}U^{\check{B}} = 0$, may be called the relativistic "stress or Cauchy" tensor and contains the "stress-strain relation" through the dependence of Z_{ij} on $S_i{}^j$ implied by the consitutive equation $e = e[\alpha, \gamma^{(M)}, G, ..]$ of the material.

For an isotropic elastic material we get

$$Z_{ij} = V\left[pG_{ij}^{-1} + B\gamma_{ij}^{(M)} + CG_{ij}\right], \tag{7.14}$$

where $p = -\frac{\partial e}{\partial V}$, $B = \frac{2}{V}\frac{\partial e}{\partial h}$, $C = \frac{2}{V}\frac{\partial e}{\partial q}$ and we get

$$de(V, h, q) = -pdV + \frac{1}{2}VBdh + \frac{1}{2}VCdq. \tag{7.15}$$

The response parameters describe the reaction of the material to the strain: p is the "isotropic stress", while B and C give the anisotropic response as in non-relativistic elesticity [perfect fluids have $e = e(V)$, $B = C = 0$, $Z_{ij} = VpG_{ij}^{-1}$ and $de(V) = -pdV$ is the Pascal law].

See Ref. [35] for a different description of relativistic Hooke law in linear elasticity: there is a 4-dimensional deformation tensor $S_{\mu\nu} = \frac{1}{2}({}^4\nabla_\mu\xi_\nu + {}^4\nabla_\nu\xi_\mu)$ and the constitutive equations of the material are given in the form $T^{\mu\nu} = C^{(\mu\nu)(\alpha\beta)}S_{\alpha\beta}$.

In Ref. [6] the theory is also extended to the thermodynamics of isentropic flows (no heat conductivity). The function e is considered also as a function of entropy $S = S(\alpha)$ and $de = \frac{1}{2}Z_{ij}dG^{ij}$ is generalized to

$$de = \frac{1}{2}Z_{ij}dG^{ij} - SdT. \tag{7.16}$$

Then e is replaced with the Helmholtz free energy $f = e - TS$ so to obtain

$$df = \frac{1}{2}Z_{ij}dG^{ij} - SdT, \tag{7.17}$$

[for perfect fluids we get $de(V, S) = -pdV + TdS$, $df(V, T) = -pdV - SdT$]. This suggests to consider the temperature T as a strain and the entropy S as the corresponding stress and to introduce an extra scalar field $\alpha^\tau(\tau, \vec\sigma)$ so that $T = const.U^{\check{A}}\partial_{\check{A}}\alpha^\tau$. The potential $\alpha^\tau(\tau, \vec\sigma)$ has the microscopic interpretation as the retardation of the proper time of the molecules with respect to the physical time calculated over averaged spacetime trajectories of the idealized continuum material.

In this case the action principle becomes $S = -\int d\tau d^3\sigma \left[N\sqrt{\gamma}nf(G, T)\right](\tau, \vec\sigma)$ and one gets the conserved energy-momentum tensor $T_{\check{A}\check{B}} = N\sqrt{\gamma}n[(f + TS)U_{\check{A}}U_{\check{B}} + Z_{\check{A}\check{B}}]$.

In all these cases one can develop the rest-frame instant form just in the same way as it was done in Section II and III for perfect fluids, even if it is not possible to obtain a closed form of the invariant mass.

VIII. CONCLUSIONS.

In this paper we have studied the Hamiltonian description in Minkowski spacetime associated with an action principle for perfect fluids with an equation of state of the form $\rho = \rho(n, s)$ given in Ref. [1], in which the fluid is descrbed only in terms of Lagrangian coordinates.

This action principle can be reformulated on arbitrary spacelike hypersurfaces embedded in Minkowski spacetime (covariant 3+1 splitting of Minkowski spacetime) along the lines of Refs. [10,11]. At the Hamiltonian leve the canonical Hamiltonian vanishes and the theory is governed by four first class constraints $\mathcal{H}^\mu(\tau, \vec{\sigma}) \approx 0$ implying the independence of the description from the choice of the 3+1 splitting of Minkowski spacetime.

These constraints can be obtained in closed form only for the 'dust' and for the 'photon gas'. For other types of perfect fluids one needs numerical calculations. After the inclusion of the coupling to the gravitational field one could begin to think to formulate Hamiltonian numerical gravity with only physical degrees of freedom and hyperbolic Hamilton equations for them [like the form (2.50) of the relativistic Euler equations for the dust].

After the canonical reduction to 3+1 splittings whose leaves are spacelike hyperplanes, we consider all the configurations of the perfect fluid whose conserved 4-momentum is timelike. For each of these configurations we can select the special foliation of Minkowski spacetime with spacelike hyperplanes orthogonal to the 4-momentum of the configuration,

This gives rise to the "Wigner-covariant rest-frame instant form of dynamics" [10,11] for the perfect fluids. After a discussion of the "external" and "internal" centers of mass and realizations of the Poincaré algebra, rest-frame Dixon's Cartesian multipoles [20] of the perfect fluid are studied.

It is also shown that the formulation of non-dissipative elastic materials of Ref. [6], based on the use of Lagrangian coordinates, allows to get the rest-frame instant form for these materials too.

Finally it is shown how to make the coupling to the gravitational field by giving the ADM action for the perfect fluid in tetrad gravity. Now it becomes possible to study the canonical reduction of tetrad gravity with the perfect fluids as matter along the lines of Refs. [25–27].

APPENDIX A: RELATIVISTIC PERFECT FLUIDS.

As in Ref. [1] let us consider a perfect fluid in a curved spacetime M^4 with unit 4-velocity vector field $U^\mu(z)$, Lagrangian coordinates $\tilde{\alpha}^i(z)$, particle number density $n(z)$, energy density $\rho(z)$, entropy per particle $s(z)$, pressure $p(z)$, temperature $T(z)$. Let $J^\mu(z) = \sqrt{^4g(z)}n(z)U^\mu(z)$ the densitized particle number flux vector field, so that we have $n = \sqrt{\epsilon\,^4g_{\mu\nu}J^\mu J^\nu}/\sqrt{^4g}$. Other local thermodynamical variables are the chemical potential or specific enthalpy (the energy per particle required to inject a small amount of fluid into a fluid sample, keeping the sample volume and the entropy per particle s constant)

$$\mu = \frac{1}{n}(\rho + p), \tag{A1}$$

the physical free energy (the injection energy at a constant number density n and constant total entropy)

$$a = \frac{\rho}{n} - Ts, \tag{A2}$$

and the chemical free energy (the injection energy at constant volume and constant total entropy)

$$f = \frac{1}{n}(\rho + p) - Ts = \mu - Ts. \tag{A3}$$

Since the local expression of the first law of thermodynamics is

$$d\rho = \mu dn + nTds, \quad or \quad dp = nd\mu - nTds, \quad or \quad d(na) = fdn - nsdT, \tag{A4}$$

an equation of state for a perfect fluid may be given in one of the following forms

$$\rho = \rho(n,s), \quad or \quad p = p(\mu, s), \quad or \quad a = a(n, T). \tag{A5}$$

By definition, the stress-energy-momentum tensor for a perfect fluid is

$$T^{\mu\nu} = -\epsilon\,\rho U^\mu U^\nu + p(^4g^{\mu\nu} - \epsilon U^\mu U^\nu) = -\epsilon(\rho + p)U^\mu U^\nu + p\,^4g^{\mu\nu}, \tag{A6}$$

and its equations of motion are

$$T^{\mu\nu}{}_{;\nu} = 0, \qquad (nU^\mu)_{;\mu} = \frac{1}{\sqrt{^4g}}\partial_\mu J^\mu = 0. \tag{A7}$$

As shown in Ref. [1] an action functional for a perfect fluid depending upon $J^\mu(z)$, $^4g_{\mu\nu}(z)$, $s(z)$ and $\tilde{\alpha}^i(z)$ requires the introduction of the following Lagrange multipliers to implement all the required properties:

i) $\theta(z)$: it is a scalar field named 'thermasy'; it is interpreted as a potential for the fluid temperature $T = \frac{1}{n}\frac{\partial\rho}{\partial s}|_n$. In the Lagrangian it is interpreted as a Lagrange multiplier for implementing the "entropy exchange constraint" $(sJ^\mu)_{,\mu} = 0$.

ii) $\varphi(z)$: it is a scalar field; it is interpreted as a potential for the chemical free energy f. In the Lagrangian it is interpreted as a Lagrange multipliers for the "particle number conservation constraint" $J^\mu{}_{,\mu} = 0$.

iii) $\beta_i(z)$: they are three scalar fields; in the Lagrangian they are interpreted as Lagrange multipliers for the "constraint" $\tilde{\alpha}^i{}_{,\mu} J^\mu = 0$ that restricts the fluid 4-velocity vector to be directed along the flow lines $\tilde{\alpha}^i = const$.

Given an arbitrary equation of state of the type $\rho = \rho(n, s)$, the action functional is

$$S[{}^4g_{\mu\nu}, J^\mu, s, \tilde{\alpha}, \varphi, \theta, \beta_i] = \int d^4z \{ -\sqrt{{}^4g}\rho(\frac{|J|}{\sqrt{{}^4g}}, s) +$$
$$+ J^\mu[\partial_\mu\varphi + s\partial_\mu\theta + \beta_i\partial_\mu\tilde{\alpha}^i] \}. \tag{A8}$$

By varying the 4-metric we get the standard stress-energy-momentum tensor

$$T^{\mu\nu} = \frac{2}{\sqrt{{}^4g}}\frac{\delta S}{\delta\,{}^4g_{\mu\nu}} = -\epsilon\rho U^\mu U^\nu + p({}^4g^{\mu\nu} - \epsilon U^\mu U^\nu) = -\epsilon(\rho + p)U^\mu U^\nu + p\,{}^4g^{\mu\nu}, \tag{A9}$$

where the pressure is given by

$$p = n\frac{\partial\rho}{\partial n}|_s - \rho. \tag{A10}$$

The Euler-Lagrange equations for the fluid motion are

$$\frac{\delta S}{\delta J^\mu} = \mu U_\mu + \partial_\mu\varphi + s\partial_\mu\theta + \beta_i\partial_\mu\tilde{\alpha}^i = 0,$$
$$\frac{\delta S}{\delta\varphi} = -\partial_\mu J^\mu = 0,$$
$$\frac{\delta S}{\delta\theta} = -\partial_\mu(sJ^\mu) = 0,$$
$$\frac{\delta S}{\delta s} = -\sqrt{{}^4g}\frac{\partial\rho}{\partial s} + J^\mu\partial_\mu\theta = 0,$$
$$\frac{\delta S}{\delta\tilde{\alpha}^i} = -\partial_\mu(\beta_i J^\mu) = 0,$$
$$\frac{\delta S}{\delta\beta_i} = J^\mu\partial_\mu\tilde{\alpha}^i = 0. \tag{A11}$$

The second equation is the particle number conservation, the third one the entropy exchange constraint and the last one restricts the fluid 4-velocity vector to be directed along the flow lines $\tilde{\alpha}^i = const.$. The first equation gives the Clebsch or velocity-potential representation of the 4-velocity U_μ (the scalar fields in this representation are called Clebsch or velocity potentials). The fifth equations imply the constancy of the β_i's along the fluid flow lines, so that these Lagrange multipliers can be expressed as a function of the Lagrangian coordinates. The fourth equation, after a comparison with the first law of thermodynamics, leads to the identification $T = U^\mu\partial_\mu\theta = \frac{1}{n}\frac{\partial\rho}{\partial s}|_n$ for the fluid temperature.

Moreover, one can show that the Euler-Lagrange equations imply the conservation of the stress-energy-momentum tensor $T^{\mu\nu}{}_{;\nu} = 0$. This equations can be split in the projection

along the fluid flow lines and in the one orthogonal to them:

i) The projection along the fluid flow lines plus the particle number conservation give $U_\mu T^{\mu\nu}{}_{;\nu} = -\frac{\partial\rho}{\partial s}U^\mu\partial_\mu s = 0$, which is verified due to the entropy exchange constraint. Therefore, the fluid flow is locally adiabatic, that is the entropy per particle along the fluid flow lines is conserved.

ii) The projection orthogonal to the fluid flow lines gives the Euler equations, relating the fluid acceleration to the gradient of pressure

$$({}^4g_{\mu\nu} - \epsilon U_\mu U_\nu)T^{\nu\alpha}{}_{;\alpha} = -\epsilon(\rho + p)U_{\mu;\nu}U^\nu - (\delta^\nu_\mu - \epsilon U_\mu U^\nu)\partial_\nu p. \tag{A12}$$

By using $p = n\frac{\partial\rho}{\partial n}|_s - \rho$, it is shown in Ref. [1] that these equations can be rewritten as

$$2(\mu U_{[\mu})_{;\nu]}U^\nu = -\epsilon(\delta^\nu_\mu - U_\mu U^\nu)\frac{1}{n}\frac{\partial\rho}{\partial s}|_n\partial_\nu s. \tag{A13}$$

The use of the entropy exchange constraint allows the rewrite the equations in the form

$$2V_{[\mu;\nu]}U^\nu = T\partial_\mu s, \tag{A14}$$

where $V_\mu = \mu U_\mu$ is the Taub current (important for the description of circulation and vorticity), which can be identified with the 4-momentum per particle of a small amount of fluid to be injected in a larger sample of fluid without changing the total fluid volume or the entropy per particle. Now from the Euler-Lagrange we get

$$2V_{[\mu;\nu]}U^\nu = -2(\partial_{[\mu}\varphi + s\partial_{[\mu}\theta + \beta_i\partial_{[\mu}\tilde\alpha^i)_{;\nu]}U^\nu = (s\partial_{[\mu}\theta)_{;\nu]}U^\nu = T\partial_\mu s, \tag{A15}$$

and this result implies the validity of the Euler equations.

In the non-relativistic limit $(nU^\mu)_{;\mu} = 0$, $T^{\mu\nu}{}_{;\nu} = 0$ become the particle number (or mass) conservation law, the entropy conservation law and the Euler-Newton equations. See Refs. [23,30] for the post-Newtonian approximation.

We refer to Ref. [1] for the complete discussion. The previous action has the advantage on other actions that the canonical momenta conjugate to φ and θ are the particle number density and entropy density seen by Eulerian observers at rest in space. The action evaluated on the solutions of the equations of motion is $\int d^4z\sqrt{{}^4g(z)}p(z)$.

In Ref. [1] there is a study of a special class of global Noether symmetries of this action associated with arbitrary functions $F(\tilde\alpha, \beta_i, s)$. It is shown that for each F there is a conservation equation $\partial_\mu(FJ^\mu) = 0$ and a Noether charge $Q[F] = \int_\Sigma d^3\sigma\sqrt\gamma n\,(\epsilon l_\mu U^\mu)F(\tilde\alpha, \beta_i, s)$ [Σ is a spacelike hypersurface with future pointing unit normal l_μ and with a 3-metric with determinant $\sqrt\gamma$]. For $F = 1$ inside a volume V in Σ we get the conservation of particle number within a flow tube defined by the bundle of flow lines contained in the volume V. The factor $\epsilon l_\mu U^\mu$ is the relativistic 'gamma factor' characterizing a boost from the Lagrangian observers with 4-velocity U^μ to the Eulerian observers with 4-velocity l^μ; thus $n(\epsilon l_\mu U^\mu)$ is the particle number density as seen from the Eulerian observers. These symmetries describe the changes of Lagrangian coordinates $\tilde\alpha^i$ and the fact that both the Lagrange multipliers φ and θ are constant along each flow line (so that it is possible to transform any solution

to the fluid equations of motion into a solution with $\varphi = \theta = 0$ on any given spacelike hypersurface).

However, the Hamiltonian formulation associated with this action is not trivial, because the many redundant variables present in it give rise to many first and second class constraints. In particular we get:

1) second class constraints:

A) $\pi_{J^\tau} \approx 0$, $J^\tau - \pi_\varphi \approx 0$;

B) $\pi_s \approx 0$, $sJ^\tau - \pi_\theta \approx 0$;

C) $\pi_{\beta_s} \approx 0$, $\beta_s J^\tau - \pi_{\alpha^s} \approx 0$.

2) first class constraints: $\pi_{J^r} \approx 0$, so that the J^r's are gauge variables.

Therefore the physical variables are the five pairs: φ, π_φ; θ, π_θ; $\tilde{\alpha}^i$, π_{α^r} and one could study the associated canonical reduction.

In Ref. [1] [see its rich bibliography for the references] there is a systematic study of the action principles associated to the three types of equations of state present in the literature, first by using the Clebsch potentials and the associated Lagrange multipliers, then only in terms of the Lagrangian coordinates by inserting the solution of some of the Euler-Lagrange equations in the original action and eventually by adding surface terms.

1) Equation of state $\rho = \rho(n, s)$. One has the action

$$S[n, U^\mu, \varphi, \theta, s, \tilde{\alpha}^r, \beta_r; {}^4g_{\mu\nu}] = -\int d^4x \sqrt{{}^4g} \Big[\rho(n, s) - nU^\mu(\partial_\mu \varphi - \theta \partial_\mu s + \beta_r \partial_\mu \tilde{\alpha}^r) \Big] \quad \text{(A16)}$$

If one knows $s = s(\tilde{\alpha}^r)$ and $J^\mu = J^\mu(\tilde{\alpha}^r) = -\sqrt{{}^4g}\epsilon^{\mu\nu\rho\sigma}\partial_\nu \tilde{\alpha}^1 \partial_\rho \tilde{\alpha}^2 \partial_\sigma \tilde{\alpha}^3 \eta_{123}(\tilde{\alpha}^r)$, one can define $\tilde{S} = S - \int d^4x \partial_\mu [(\varphi + s\theta)J^\mu]$, and one can show that it has the form

$$\tilde{S} = \tilde{S}[\tilde{\alpha}^r] = -\int d^4x \sqrt{{}^4g} \rho\Big(\frac{|J|}{\sqrt{{}^4g}}, s\Big). \quad \text{(A17)}$$

2) Equation of state: $p = p(\mu, s)$ [$V^\mu = \mu U^\mu$ Taub vector]

$$S_{(p)} = s_{(p)}[V^\mu, \varphi, \theta, s, \tilde{\alpha}^r, \beta_r; {}^4g_{\mu\nu}] =$$
$$= \int d^4x \sqrt{{}^4g} \Big[p(\mu, s) - \frac{\partial p}{\partial \mu}\Big(|V| - \frac{V^\mu}{|V|}(\partial_\mu \varphi + s\partial_\mu \theta + \beta_r \partial_\mu \tilde{\alpha}^r)\Big) \Big] \quad \text{(A18)}$$

or by using one of its EL equations $V_\mu \overset{\circ}{=} -(\partial_\mu \varphi + s\partial_\mu \theta + \beta_r \partial_\mu \tilde{\alpha}^r)$ to eliminate V^μ one gets Schutz's action [μ determined by $\mu^2 = -V^\mu V_\mu$]

$$\tilde{S}_{(p)}[\varphi, \theta, s, \tilde{\alpha}^r, \beta_r; {}^4g_{\mu\nu}] = \int d^4x \sqrt{{}^4g} p(\mu, s) \quad \text{(A19)}$$

3) Equation of state $a = a(n, T)$. The action is

$$S_{(a)}[J^\mu, \varphi, \theta, \tilde{\alpha}^r, \beta_r; {}^4g_{\mu\nu}] = \int d^4x \Big[|J|a\Big(\frac{|J|}{\sqrt{{}^4g}}, \partial_\mu \theta J^\mu\Big) - J^\mu(\partial_\mu \varphi + \beta_r \partial_\mu \tilde{\alpha}^r) \Big] \quad \text{(A20)}$$

or $\tilde{S}_{(a)}[\varphi, \theta, s, \tilde{\alpha}^r, \beta_r] = S_{(a)} - \int d^4x \Big[|J|a\Big(\frac{|J|}{\sqrt{{}^4g}}, \frac{J^\mu}{|J|}\partial_\mu \theta\Big) \Big]$.

At the end of Ref. [1] there is the action for "isentropic" fluids and for their particular case of a "dust" (used in Ref. [9] as a reference fluid in canonical gravity).

The isentropic fluids have equation of state $a(n,T) = \frac{\rho(n)}{n} - sT$ with $s = const.$ (constant value of the entropy per particle). By introducing $\varphi' = \varphi + s\theta$, the action can be written in the form

$$S_{(isentrpic)}[J^\mu, \varphi', \tilde{\alpha}^r, \beta_r, {}^4g^{\mu\nu}] = \int d^4x \left[-\sqrt{{}^4g}\rho\left(\frac{|J|}{\sqrt{{}^4g}}\right) + J^\mu(\partial_\mu\varphi' + \beta_r\partial_\mu\tilde{\alpha}^r) \right] \qquad (A21)$$

or

$$\tilde{S}_{(isentropic)}[\tilde{\alpha}^r; {}^4g_{\mu\nu}] = -\int d^4x\sqrt{{}^4g}\rho\left(\frac{|J|}{\sqrt{{}^4g}}\right) \qquad (A22)$$

The dust has equation of state $\rho(n) = \mu n$, namely $a(n,T) = \mu - sT$ so that we get zero pressure $p = n\frac{\partial\rho}{\partial n} - \rho = 0$. Again with $\varphi' = \varphi + s\theta$ the action becomes

$$S_{(dust)}J^\mu, \varphi', \tilde{\alpha}^r, \beta_r, {}^4g^{\mu\nu}] = \int d^4x \left[-\mu|J| + J^\mu(\partial_\mu\varphi' + \beta_r\partial_\mu\tilde{\alpha}^r) \right] \qquad (A23)$$

or with $U_\mu = -\frac{1}{\mu}(\partial_\mu\varphi' + \beta_r\partial_\mu\tilde{\alpha}^r)$ [In Ref. [9]: $M = \mu n$ rest mass (energy) density and $T = \varphi'/\mu$, $W_r = -\beta_r$, $Z^r = \tilde{\alpha}^r$; $U_\mu = -\partial_\mu T + W_r\partial_\mu Z^r$]

$$S'_{(dust)}[T, Z^r, M, W_r; {}^4g_{\mu\nu}] = -\frac{1}{2}\int d^4x\sqrt{{}^4g}(\mu n)\left(U_\mu\,{}^4g^{\mu\nu}U_\nu - \epsilon\right), \qquad (A24)$$

or

$$\tilde{S}_{(dust)}[\tilde{\alpha}^r; {}^4g_{\mu\nu}] = -\int d^4x\mu|J| \qquad (A25)$$

In Ref. [9] there is a study of the action (A24) since the dust is used as a reference fluid in general relativity. At the Hamiltonian level one gets:
i) 3 pairs of second class constraints $[\pi^r_{\vec{W}}(\tau, \vec{\sigma}) \approx 0, \pi_{\vec{Z}\,r}(\tau, \vec{\sigma}) - W_r(\tau, \vec{\sigma})\pi_T(\tau, \vec{\sigma}) \approx 0]$, which allow the elimination of $W_r(\tau, \vec{\sigma})$ and $\pi^r_{\vec{W}}(\tau, \vec{\sigma})$;
ii) a pair of second class constraints $[\ \pi_M(\tau, \vec{\sigma}) \approx 0$ plus the secondary $M(\tau, \vec{\sigma}) - \frac{\pi_T^2}{\sqrt{\gamma}\sqrt{\pi_T^2 + {}^3g^{rs}(\pi_T\partial_r T + \pi_{\vec{Z}\,u}\partial_r Z^u)(\pi_T\partial_s T + \pi_{\vec{Z}\,v}\partial_s Z^v)}}(\tau, \vec{\sigma}) \approx 0]$, which allow the elimination of M, π_M.

APPENDIX B: COVARIANT RELATIVISTIC THERMODYNAMICS OF EQUILIBRIUM AND NON-EQUILIBRIUM.

In this Appendix we shall collect some results on relativistic fluids which are well known but scattered in the specialized literature. We shall use essentially Ref. [2], which has to be consulted for the relevant bibliography. See also Ref. [36].

Firstly we remind some notions of covariant thermodynamics of equilibrium.

Let us remember that given the stress-energy-momentum tensor of a continuous medium $T^{\mu\nu}$, the densities of energy and momentum are T^{oo} and $c^{-1}T^{ro}$ respectively [so that $dP^\mu = c^{-1}\eta T^{\mu\nu}d\Sigma_\nu$ is the 4-momentum that crosses the 3-area element $d\Sigma_\nu$ in the sense of its normal ($\eta = -1$ if the normal is spacelike, $\eta = +1$ if it is timelike)]; instead, cT^{or} is the energy flux in the positive r direction, while T^{rs} is the r component of the stress in the plane perpendicular to the s direction (a pressure, if it is positive). A local observer with timelike 4-velocity u^μ ($u^2 = \epsilon c^2$) will measure energy density $c^{-2}T^{\mu\nu}u_\mu u_\nu$ and energy flux $\epsilon T^{\mu\nu}u_\mu n_\nu$ along the direction of a unit vector n^μ in his rest frame.

For a fluid at thermal equilibrium with $T^{\mu\nu} = \rho U^\mu U^\nu - \epsilon\frac{p}{c^2}(^4g^{\mu\nu} - \epsilon U^\mu U^\nu)$ [U^μ is the hydrodynamical 4-velocity of the fluid] with particle number density n, specific volume $V = \frac{1}{n}$ and entropy per particle $s = \frac{k_B S}{n}$ (k_B is Boltzmann's constant) in its rest frame, the energy density is

$$\rho c^2 = n(mc^2 + e), \tag{B1}$$

where e is the mean internal (thermal plus chemical) energy per particle and m is particle's rest mass.

From a non-relativistic point of view, by writing the equation of state in the form $s = s(e, V)$ the temperature and the pressure emerge as partial derivatives from the first law of thermodynamics in the form (Gibbs equation)

$$ds(e, V) = \frac{1}{T}(de + pdV). \tag{B2}$$

If $\mu_{clas} = e + pV - Ts$ is the non-relativistic chemical potential per particle, its relativistic version is

$$\mu' = mc^2 + \mu_{class} = \mu - Ts, \tag{B3}$$

[$\mu = \frac{\rho c^2 + p}{n}$ is the specific enthalpy, also called chemical potential as in Appendix A] and we get

$$\mu' n = \rho c^2 + p - nTs = \rho c^2 + p - k_B TS,$$
$$k_B TdS = d(\rho c^2) - \mu' dn = d(\rho c^2) - (\mu - Ts)dn, \qquad or$$
$$d(\rho c^2) = \mu' dn + Td(ns) = \mu dn + nTds. \tag{B4}$$

By introducing the "thermal potential" $\alpha = \frac{\mu'}{k_B T} = \frac{\mu - Ts}{k_B T}$ and the inverse temperature $\beta = \frac{c^2}{k_B T}$, these two equations take the form

$$S = \frac{ns}{k_B} = \beta(\rho + \frac{p}{c^2}) - \alpha n,$$

$$dS = \beta d\rho - \alpha dn. \tag{B5}$$

Let us remark that in Refs. [23,29,37] one uses different notations, some of which are given in the following equation (in Ref. [23] ρ is denoted e and ρ' is denoted r)

$$\rho + \frac{p}{c^2} = n(mc^2 + e) + \frac{p}{c^2} = \rho' h = \rho'(c^2 + e' + \frac{p}{c^2 \rho'}) = \rho'(c^2 + h'), \tag{B6}$$

where $\rho' = nm$ is the rest-mass density [$r_* = \sqrt{^4g}\rho'$ is called the coordinate rest-mass density] and $e' = e/m$ is the specific internal energy [so that $\rho' h$ is the "effective inertial mass of the fluid; in the post-Newtonian approximation of Ref. [23] it is shown that $\sigma = c^{-2}(T^{oo} + \sum_s T^{ss}) + O(c^{-4}) = c^{-2}\sqrt{^4g}(-T_o^o + T_s^s) + O(c^{-4})$ has the interpretation of equality of the "passive" and the "active" gravitational mass]. For the specific enthalpy or chemical potential we get [$\mu/m = h = c^2 + h'$ is called enthalpy]

$$\mu = \frac{1}{n}(\rho + \frac{p}{c^2}) = \frac{m}{\rho'}(\rho + \frac{p}{c^2}) = mh = m(c^2 + h'). \tag{B7}$$

See Ref. [29] for a richer table of conversion of notations.

Relativistically, we must consider, besides the stress-energy-momentum tensor $T^{\mu\nu}$ and the associated 4-momentum $P^\mu = \int_V d^3\Sigma_\nu T^{\mu\nu}$, a particle flux density n^μ (one n_a^μ for each constituent a of the system) and the entropy flux density s^μ. At thermal equilibrium all these a priori unrelated 4-vectors must all be parallel to the hydrodynamical 4-velocity

$$n^\mu = nU^\mu \quad , \quad s^\mu = sU^\mu, \quad P^\mu = PU^\mu, \tag{B8}$$

Analogously, we have $V^\mu = VU^\mu$ ($V = 1/n$ is the specific volume), $\beta^\mu = \beta U^\mu = \frac{c^2}{k_B T}U^\mu$ [a related 4-vector is the equilibrium parameter 4-vector $i^\mu = \mu'\beta^\mu$].

Since $\epsilon U_\mu T^{\mu\nu} = \rho U^\nu$, we get the final manifestly covariant form of the previous two equations (now the hydrodynamical 4-velocity is considered as an extra thermodynamical variable)

$$S^\mu = SU^\mu = \frac{ns}{k_B}U^\mu = \frac{p}{c^2}\beta^\mu - \alpha n^\mu - \epsilon \beta_\nu T^{\nu\mu},$$

$$dS^\mu = -\alpha dn^\mu - \epsilon \beta_\nu dT^{\nu\mu}. \tag{B9}$$

Global thermal equilibrium imposes

$$\partial_\mu \alpha = \partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0. \tag{B10}$$

As a consequence we get

$$d(\frac{p}{c^2}\beta^\mu) = n^\mu d\alpha + \epsilon T^{\nu\mu} d\beta_\nu, \tag{B11}$$

namely the basic variables n^μ, $T^{\nu\mu}$ and S^μ can all be generated from partial derivatives of the "fugacity" 4-vector (or "thermodynamical potential")

$$\phi^\mu(\alpha, \beta_\lambda) = \frac{p}{c^2}\beta^\mu,$$

$$n^\mu = \frac{\partial \phi^\mu}{\partial \alpha}, \qquad T^{\nu\mu}_{(mat)} = \frac{\partial \phi^\mu}{\partial \beta_\nu}, \qquad S^\mu = \phi^\mu - \alpha n^\mu - \beta_\nu T^{\nu\mu}_{(mat)}, \qquad (B12)$$

once the equation of state is known. Here $T^{\nu\mu}_{(mat)}$ is the canonical or material (in general non symmetric) stress tensor, ensuring that reversible flows of field energy are not accompanied by an entropy flux.

This final form remains valid (at least to first order in deviations) for "states that deviate from equilibrium", when the 4-vectors S^μ, n^μ,... are no more parallel; the extra information in this equation is precisely the standard linear relation between entropy flux and heat flux. The second law of thermodynamics for relativistic systems is $\partial_\mu S^\mu \geq 0$, which becomes a strict equality in equilibrium.

The fugacity 4-vector ϕ^μ is evaluated by using the covariant relativistic statistical theory for thermal equilibrium [2] starting from a grand canonical ensemble with density matrix $\hat\rho$ by maximizing the entropy $S = -Tr(\hat\rho ln\, \hat\rho)$ subject to the constraints $Tr\hat\rho = 1$, $Tr(\hat\rho\hat n) = n$, $Tr(\hat\rho\hat P^\lambda) = P^\lambda$: this gives (in the large volume limit)

$$\hat\rho = Z^{-1}e^{\alpha\hat n + \beta_\mu \hat P^\mu},$$
$$\text{with}$$
$$ln\, Z = \int_{\triangle\Sigma} \epsilon\phi^\mu d\Sigma_\mu, \qquad n = \int_{\triangle\Sigma} \epsilon n^\mu d\Sigma_\mu, \qquad P^\mu = \int_{\triangle\Sigma} \epsilon T^{\mu\nu}_{(mat)} d\Sigma_\nu, \qquad (B13)$$

[it is assumed that the members of the ensemble are small (macroscopic) subregions of one extended body in thermal equilibrium, whose worldtubes intersect an arbitrary spacelike hypersurface in small 3-areas $\triangle\Sigma$]. Therefore, one has to find the grand canonical partition function

$$Z(V_\mu, \beta_\mu, i_\mu) = \sum_n e^{i_\mu n^\mu} Q_n(V_\mu, \beta_\mu), \qquad where \quad Q_n(V_\mu, \beta_\mu) = \int_{V_\mu} d\sigma_n(q, p)e^{-\beta_\mu P^\mu}, \quad (B14)$$

is the canonical partition function for fixed volume V_μ and $d\sigma_n$ is the invariant microcanonical density of states. For an ideal Boltzmann gas of N free particles of mass m [see Section III for its equation of state] it is

$$d\sigma_n(p, m) = \frac{1}{N!} \int \delta^4(P - \sum_{i=1}^N p_i) \prod_{i=1}^N 2V_\mu p_i^\mu \theta(p_i^o)\delta(p_i^2 - \epsilon m^2)d^4 p_i. \qquad (B15)$$

Following Ref. [38] [using a certain type of gauge fixings to the first class constraints $p_i^2 - \epsilon m^2 \approx 0$] in Ref. [10] Q_n was evaluated in the rest-frame instant form on the Wigner hyperplane (this method can be extended to a gas of molecules, which are N-body bound states):

$$Q_n = \frac{1}{N!}\Big[\frac{V m^2}{2\pi^2 \beta}K_2(m\beta)\Big]^N. \qquad (B16)$$

The same results may be obtained by starting from the covariant relativistic kinetic theory of gas [see Ref. [39,40]; in Ref. [2] there is a short review] whose particles interact

only by collisions by using Synge's invariant distribution function $N(q,p)$ [41] [the number of particle worldlines with momenta in the range $(p_\mu, d\omega)$ that cross a target 3-area $d\Sigma_\mu$ in M^4 in the direction of its normal is given by $dN = N(q,p)d\omega\eta v^\mu d\Sigma_\mu$ ($= Nd^3qd^3p$ for the 3-space $q^o = const.$); $d\omega = d^3p/v^o\sqrt{^4g}$ is the invariant element of 3-area on the mass-shell]. One arrives at a transport equation for $N, \frac{dN}{d\tau} = \nabla_\mu(Nv^\mu)$ [v^μ is the particle velocity obtained from the Hamilton equation implied by the one-particle Hamiltonian $H = \sqrt{\epsilon\,^4g^{\mu\nu}(q)p_\mu p_\nu} = m$ (it is the energy after the gauge fixing $q^o \approx \tau$ to the first class contraint $^4g^{\mu\nu}(q)p_\mu p_\nu - \epsilon m^2 \approx 0$); ∇_μ is the covariant gradient holding the 4-vector p_μ (not its components) fixed] with a "collision term" $C[N]$ describing the collisions; for a dilute simple gas dominated by binary collisions one arrives at the Boltzmann equation [for $C[N] = 0$ one solution is the relativistic version of the Maxwell-Boltzmann distribution function, i.e. the classical Jüttner-Synge one $N = const.e^{-\beta_\mu P^\mu}/4\pi m^2 K_2(m\beta)$ for the Boltzmann gas [41]]. The H-theorem [$\nabla_\mu S^\mu \geq 0$, where $S^\mu(q) = -\int[N ln(Nh^3) - N]v^\mu d\omega$ is the entropy flux] and the results at thermal equilibrium emerge [from the balance law $\nabla_\mu(\int Nf v^\mu d\omega) = \int fC[N]d\omega$ (f is an arbitrary tensorial function) one can deduce the conservation laws $\nabla_\mu n^\mu = \nabla_\mu T^{\nu\mu} = 0$, where $n^\mu = \int Nv^\mu d\omega$, $T_\nu{}^\mu = \int Np_\nu v^\mu d\omega$; the vanishing of entropy production at local thermal equilibrium gives $N_{eq}(q,p) = h^{-3}e^{\alpha(q)+\beta_\nu(q)P^\nu}$ in the case of Boltzmann statistic and one gets ($U^\mu = \beta^\mu/\beta$) $n_{eq}^\mu = \int N_{eq}v^\mu d\omega = nU^\mu$, $T_{eq}^{\nu\mu} = \rho U^\nu U^\mu - \epsilon p(^4g^{\nu\mu} - \epsilon U^\nu U^\mu)$, $S_{eq}^\mu = p\beta^\mu - \alpha n_{eq}^\mu - \beta_\nu T_{eq}^{\mu\nu}$; one obtains the equations for the Boltzmann ideal gas given in Section III].

One can study the small deviations from thermal equilibrium [$N = N_{eq}(1+f)$, where N_{eq} is an arbitrary local equilibrium distribution] with the linearized Boltzmann equation and then by using either the Chapman-Enskog ansatz of quasi-stationarity of small deviations (this ignores the gradients of f and gives the standard Landau-Lifshitz and Eckart phenomenological laws; one gets Fourier equation for heat conduction and the Navier-Stokes equation for the bulk and shear stresses; however one has parabolic and not hyperbolic equations implying non-causal propagation) or with the Grad method in the 14-moment approximation. This method retains the gradients of f [there are 5 extra thermodynamical variables, which can be explicitly determined from 14 moments among the infinite set of moments $\int Np^\mu p^\nu p^\rho...d^4p$ of kinetic theory; no extra auxiliary state variables are introduced to specify a non-equilibrium state besides $T^{\mu\nu}$, n^μ, S^μ] and gives phenomenological laws which are the kinetic equivalent of Müller extended thermodynamics and its various developments; now the equations are hyperbolic, there is no causality problem but there are problems with shock waves. See Ref. [2] for the bibliography and for a review of the non-equilibrium phenomenological laws (see also Ref. [42]) of Eckart, Landau-Lifshitz, of the various formulations of extended thermodynamics, of non-local thermodynamics.

While in Ref. [43] it is said that the difference between causal hyperbolic theories and acausual parabolic one is unobservable, in Ref. [44][see also Ref. [24]] there is a discussion of the cases in which hyperbolic theories are relevant. See also the numerical codes of Refs. [45,31,32].

In phenomenological theories the starting point are the equations $\partial_\mu T^{\mu\nu} = \partial_\mu n^\mu = 0$, $\partial_\mu S^\mu \geq 0$. There is the problem of how to define a 4-velocity and a rest-frame for a given non-equilibrium state. Another problem is how to specify a non-equilibrium state completely at the macroscopic level: a priori one could need an infinite number of auxiliary quantities (vanishing at equilibrium) and an equation of state depending on them. The basic postulate

of extended thermodynamics is the absence of such variables.

Regarding the rest frame problem there are two main solutions in the literature connected with the relativistic description of "heat flow":

i) Eckart theory. One considers a local observer in a simple fluid who is at rest with respect to the average motion of the particles: its 4-velocity $U^\mu_{(eck)}$ is parallel by definition to the particle flux n^μ, namely

$$n^\mu = n_{(eck)} U^\mu_{(eck)}. \tag{B17}$$

This local observer sees "heat flow" as a flux of energy in his rest frame:

$$\epsilon U_{(eck)\mu} T^{\mu\nu} = \rho_{(eck)} U^\mu_{(eck)} + q^\mu_{(eck)},$$
$$so\, that\, we\, get$$
$$T^{\mu\nu} = \rho_{(eck)} U^\mu_{(eck)} U^\nu_{(eck)} + q^\mu_{(eck)} U^\nu_{(eck)} + U^\mu_{(eck)} q^\nu_{(eck)} + P^{\mu\nu}_{(eck)},$$

$$P^{\mu\nu}_{(eck)} = P^{\nu\mu}_{(eck)} = \epsilon(p + \pi_{(eck)})(^4g^{\mu\nu} - \epsilon U^\mu_{(eck)} U^\nu_{(eck)}) + \pi^{\mu\nu}_{(eck)},$$

$$P^{\mu\nu}_{(eck)} U_{(eck)\nu} = q_{(eck)\mu} U^\mu_{(eck)} = 0, \qquad \pi_{(eck)\mu\nu}(^4g^{\mu\nu} - \epsilon U^\mu_{(eck)} U^\nu_{(eck)}) = 0, \tag{B18}$$

where p is the thermodynamic pressure, $\pi_{(eck)}$ the bulk viscosity and $\pi^{\mu\nu}_{(eck)}$ the shear stress.

This description has the particle conservation law $\partial_\mu n^\mu = 0$.

ii) Landau-Lifshitz theory. One considers a different observer (drifting slowly in the direction of heat flow with a 3-velocity $\vec{v}_D = \vec{q}/nmc^2$) whose 4-velocity $U^\mu_{(ll)}$ is by definition such to give a vanishing "heat flow", i.e. there is no net energy flux in his rest frame: $U^\mu_{(ll)} T^\nu_\mu n_\nu = 0$ for all vectors n_μ orthogonal to $U^\mu_{(ll)}$. This implies that $U^\mu_{(ll)}$ is the timelike eigenvector of $T^{\mu\nu}$, $T^{\mu\nu} U_{(ll)\nu} = \epsilon\rho_{(ll)} U^\mu_{(ll)}$, which is unique if $T^{\mu\nu}$ satisfies a positive energy condition. Now we get

$$T^{\mu\nu} = \rho_{(ll)} U^\mu_{(ll)} U^\nu_{(ll)} + P^{\mu\nu}_{(ll)},$$

$$P^{\mu\nu}_{(ll)} = P^{\nu\mu}_{(ll)} = \epsilon(p + \pi_{(ll)})(^4g^{\mu\nu} - \epsilon U^\mu_{(ll)} U^\nu_{(ll)}) + \pi^{\mu\nu}_{(ll)},$$

$$P^{\mu\nu}_{(ll)} U_{(ll)\nu} = 0, \qquad \pi_{(ll)\mu\nu}(^4g^{\mu\nu} - \epsilon U^\mu_{(ll)} U^\nu_{(ll)}) = 0,$$

$$n^\mu = n_{(ll)} U^\mu_{(ll)} + j^\mu_{(ll)}, \qquad j_{(ll)\mu} U^\mu_{(ll)} = 0 \qquad (\vec{j} = -n\vec{v}_D = -\vec{q}/mc^2). \tag{B19}$$

This observer in his rest frame does not see a heat flow but a particle drift. This description has the simplest form of the energy-momentum tensor.

One has $n_{(eck)} = n_{(ll)} ch\,\varphi$, $\rho_{(eck)} = \rho_{(ll)} ch^2\,\varphi + p_{(ll)} sh^2\,\varphi = \pi^{\mu\nu} j_\mu j_\nu / n^2_{(eck)}$, with $ch\,\varphi = U^\mu_{(ll)} U_{(eck)\mu}$ [the difference is a Lorentz factor $\sqrt{1 - \vec{v}^2_D/c^2}$, so that there are insignificant differences for many practical purposes if deviations from equilibrium are small]. The angle $\varphi \approx j/n \approx v_D/c \approx q/nmc^2$ is a dimensionless measure of the deviation from equilibrium [$n_{(ll)} - n_{(eck)}$ and $\rho_{(ll)} - \rho_{(eck)}$ are of order φ^2].

One can decompose $T^{\mu\nu}$, n^μ, in terms of any 4-velocity U^μ that falls within a cone of angle $\approx \varphi$ containing $U^\mu_{(eck)}$ and $U^\mu_{(ll)}$; each choice U^μ gives a particle density $n(U) = \epsilon u_\mu n^\mu$ and energy density $\rho(U) = U_\mu U_\nu T^{\mu\nu}$ which are independent of U^μ if one neglects terms of order φ^2. Therefore, one has:

i) if $S_{eq}(\rho(U), n(U))$ is the equilibrium entropy density, then $S(U) = \epsilon U_\mu S^\mu = S_{eq} + O(\varphi^2)$;

ii) if $p(U) = -\frac{\partial \rho/n}{\partial 1/n}|_{S/n}$ is the (reversible) thermodynamical pressure defined as work done in an isentropic expansion (off equilibrium this definition allows to sepate it from the bulk stress $\pi(U)$ in the stress-energy-momentum tensor) and p_{eq} is the pressure at equilibrium, then $p(U) = P_{eq}(\rho(U), n(U)) + O(\varphi^2)$.

By postulating that the covariant Gibbs relation remains valid for arbitrary infinitesimal displacements $(\delta n^\mu, \delta T^{\mu\nu}, ..)$ from an equilibrium state, one gets a covariant off-equilibrium thermodynamics based on the equation

$$S^\mu = p(\alpha, \beta)\beta^\mu - \alpha n^\mu - \beta_\nu T^{\mu\nu} - Q^\mu(\delta n^\nu, \delta T^{\nu\rho}, ..),$$
$$\nabla_\mu S^\mu = -\delta n^\mu \partial_\mu \alpha - \delta T^{\mu\nu}\nabla_\nu \beta_\mu - \nabla_\mu Q^\mu \geq 0, \tag{B20}$$

with Q^μ of second order in the displacements and α, β_μ arbitrary. At equilibrium one recovers $S^\mu_{eq} = p\beta^\mu - \alpha n^\mu_{eq} - \beta_\nu T^{\mu\nu}_{eq}$, $\nabla_\mu S^\mu_{eq} = 0$ [with $U^\mu = \beta^\mu/\beta$ and (for viscous heat-conducting fluids, but not for superfluids) $\partial_\mu \alpha = \nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu = 0$].

If we choose $\beta^\mu = U^\mu/k_B T$ parallel to n^μ of the given off-equilibrium state, we are in the "Eckart frame", $U^\mu = U^\mu_{(eck)}$, and we get

$$S = \epsilon U_{(eck)\mu} S^\mu = S_{eq} + \epsilon U_{(eck)\mu} Q^\mu,$$

$$\sigma^\mu_{(eck)} = ({}^4 g^{\mu\nu} - \epsilon U^\mu_{(eck)} U^\nu_{(eck)})S_\nu = \beta q^\mu_{(eck)} - ({}^4 g^{\mu\nu} - \epsilon U^\mu_{(eck)} U^\nu_{(eck)})Q_\nu,$$

$$q^\mu_{(eck)} = -({}^4 g^{\mu\nu} - \epsilon U^\mu_{(eck)} U^\nu_{(eck)})T^\rho_\nu U_{(eck)\rho}, \tag{B21}$$

so that to linear order we get the standard relation between entropy flux $\vec{\sigma}_{(eck)}$ and heat flux $\vec{q}_{(eck)}$

$$\vec{\sigma}_{(eck)} = \frac{\vec{q}_{(eck)}}{k_B T} + (possible\ 2nd\ order\ term). \tag{B22}$$

If we choose $U^\mu = U^\mu_{(ll)}$, the timelike eigenvector of $T^{\mu\nu}$, so that $U_{(ll)\mu} T^\mu_\nu({}^4 g^\nu_\rho - \epsilon U^\nu_{(ll)} U_{(ll)\rho}) = 0$, we are in the "Landau-Lifshitz frame" and we get

$$\sigma^\mu_{(ll)} = ({}^4 g^{\mu\nu} - \epsilon U^\mu_{(ll)} U^\nu_{(ll)})S_\nu = -\alpha j^\mu_{(ll)} - ({}^4 g^{\mu\nu} - \epsilon U^\mu_{(ll)} U^\nu_{(ll)})Q_\nu,$$

$$j^\mu_{(ll)} = ({}^4 g^{\mu\nu} - \epsilon U^\mu_{(ll)} U^\nu_{(ll)})n_\nu, \tag{B23}$$

so that at linear order we get the standard relation between entropy flux $\vec{\sigma}_{(ll)}$ and diffusive flux $\vec{j}_{(ll)}$

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$$\vec{\sigma}_{(ll)} = -\frac{\mu}{k_B T}\vec{j}_{(ll)} + (possible\ 2nd\ order\ term). \tag{B24}$$

In the Landau-Lifschitz frame heat flow and diffusion are englobed in the diffusive flux $\vec{j}_{(ll)}$ relative to the mean mass-energy flow.

The entropy inequality becomes (each term is of second order in the deviations from local equilibrium)

$$0 \le \nabla_\mu S^\mu = -\delta n^\mu \partial_\mu \alpha - \delta T^{\mu\nu}\nabla_\nu \beta_\mu - \nabla_\mu Q^\mu, \tag{B25}$$

with the fitting conditions $\delta n^\mu U_\mu = \delta T^{\mu\nu}U_\mu U_\nu = 0$, which contain all information about the viscous stresses, heat flow and diffusion in the off-equilibrium state (they are dependent on the arbitrary choice of the 4-velocity U^μ).

Once a detailed form of Q^μ is specified, linear relations between irreversible fluxes $\delta T^{\mu\nu}$, δn^μ and gradients $\nabla_{(\mu}\beta_{\nu)}$, $\partial_\mu \alpha$ follow.

A) $Q^\mu = 0$ (like in the non-relativistic case).

The spatial entropy flux $\vec{\sigma}$ is only a strictly linear function of heat flux $\vec{q}$ and diffusion flux $\vec{j}$. In this case the off-equilibrium entropy density $S = \epsilon U_\mu S^\mu$ is given by the equilibrium equation of state $S = S_{eq}(\rho, n)$. We have $0 \le \nabla_\mu S^\mu = -\delta n^\mu \partial_\mu \alpha - \delta T^{\mu\nu}\nabla_\nu \beta_\mu$ with fitting conditions $\delta n^\mu U_\mu = \delta T^{\mu\nu}U_\mu U_\nu = 0$ and with U^μ still arbitrary at first order.

A1) Landau-Lifschitz frame and theory. $U^\mu = U^\mu_{(ll)}$ is the timelike eigenvector of $T^{\mu\nu}$. This and the fitting conditions imply $\delta T^{\mu\nu}U_{(ll)\nu} = 0$. The shear and bulk stresses $\pi^{\mu\nu}_{(ll)}$, $\pi_{(ll)}$ are identified by the decomposition

$$\delta T^{\mu\nu} = \pi^{\mu\nu}_{(ll)} + \pi_{(ll)}({}^4 g^{\mu\nu} - \epsilon U^\mu_{(ll)}U^\nu_{(ll)}), \qquad \pi^{\mu\nu}_{(ll)}U_{(ll)\nu} = \pi^\mu_{(ll)\mu} = 0. \tag{B26}$$

The inequality $\nabla_\mu S^\mu \ge 0$ becomes

$$-j^\mu_{(ll)}\partial_\mu \alpha - \beta\pi^{\mu\nu}_{(ll)} < \nabla_\nu \beta_\mu > -\beta\pi_{(ll)}\nabla_\mu U^\mu_{(ll)} \ge 0, \qquad j^\mu_{(ll)} = \delta n^\mu,$$

$$< X_{\mu\nu} > = [({}^4 g^\alpha_\mu - \epsilon U^\alpha_{(ll)}U_{(ll)\mu})({}^4 g^\beta_\nu - \epsilon U^\beta_{(ll)}U_{(ll)\nu}) - \\ -\frac{1}{3}({}^4 g_{\mu\nu} - \epsilon U_{(ll)\mu}U_{(ll)\nu})({}^4 g^{\alpha\beta} - \epsilon U^\alpha_{(ll)}U^\beta_{(ll)})]X_{\alpha\beta}, \tag{B27}$$

[the $< .. >$ operation extracts the purely spatial, trace-free part of any tensor].

If the equilibrium state is isotropic (Curie's principle) and if we assume that $(j^\mu_{(ll)}, \pi^{\mu\nu}_{(ll)}, \pi_{(ll)})$ are "linear and purely local" functions of the gradients, $\nabla_\mu S^\mu \ge 0$ implies

$$j^\mu_{(ll)} = -\kappa({}^4 g^{\mu\nu} - \epsilon U^\mu_{(ll)}U^\nu_{(ll)})\partial_\mu \alpha, \quad \kappa > 0, \tag{B28}$$

[it is a mixture of Fourier's law of heat conduction and of Fick's law of diffusion, stemming from the relativistic mass-energy equivalence],

and the standard Navier-Stokes equations (ζ_S, ζ_V are shear and bulk viscosities)

$$\pi_{(ll)\mu\nu} = -2\zeta_S < \nabla_\nu \beta_\mu >, \qquad \pi_{(ll)} = \frac{1}{3}\zeta_V \nabla_\mu U^\mu_{(ll)}. \tag{B29}$$

A2) Eckart frame and theory. $U^\mu_{(eck)}$ parellel to n^μ. Now we have the fitting condition $\delta n^\mu = 0$. The heat flux appears in the decomposition of $\delta T^{\mu\nu}$ [$a_{(eck)\mu} = U^\nu_{(eck)}\nabla_\nu U_{(eck)\mu}$ is the 4-acceleration]

$$\delta T^{\mu\nu} = q^\mu_{(eck)}U^\nu_{(eck)} + U^\mu_{(eck)}q^\nu_{(eck)} + \pi^{\mu\nu}_{(eck)} + \pi_{(eck)}(^4g^{\mu\nu} - \epsilon U^\mu_{(eck)}U^\nu_{(eck)}). \tag{B30}$$

The inequality $\nabla_\mu S^\mu \geq 0$ becomes

$$q^\mu_{(eck)}(\partial_\mu \alpha - \beta a_{(eck)\mu}) - \beta(\pi^{\mu\nu}_{(eck)}\nabla_\nu U_{(eck)\mu} + \pi_{(eck)}\nabla_\mu U^\mu_{(eck)}) \geq 0. \tag{B31}$$

With the simplest assumption of linearity and locality, we obtain Fourier's law of heat conduction [it is not strictly equivalent to the Landau-Lifshitz one, because they differ by spatial gradients of the viscous stresses and the time-derivative of the heat flux]

$$q^\mu_{(eck)} = -\kappa(^4g^{\mu\nu} - \epsilon U^\mu_{(eck)}U^\nu_{(eck)})(\partial_\nu T + T\partial_\tau U_{(eck)\nu}), \tag{B32}$$

[the term depending on the acceleration is sometimes referred to as an effect of the "inertia of heat"], and the same form of the Navier-Stokes equations for $\pi^{\mu\nu}_{(eck)}$, $\pi_{(eck)}$ (they are not strictly equivalent to the Landau-Lifshitz ones, because they differ by gradients of the drift $\vec{v}_D = \vec{q}/nmc^2$).

For a simple fluid Fourier's law and Navier-Stokes equations (9 equations) and the conservation laws $\nabla_\mu T^{\mu\nu} = \nabla_\mu n^\mu = 0$ (5 equations) determine the 14 variables $T^{\mu\nu}$, n^μ from suitable initial data. However, these equations are of mixed parabolic-hyperbolic- elliptic type and, as said, one gets acausality and instability.

Kinetic theory gives

$$Q^\mu = -\frac{1}{2}\int N_{eq}f^2 p^\mu d\omega \neq 0, \tag{B33}$$

for a gas up to second order in the deviation $(N - N_{eq}) = N_{eq}f$ [$Q^\mu = 0$ requires small gradients and quasi-stationary processes]. Two alternative classes of phenomenological theories are

B) Linear non-local thermodynamics (NLT).

This theory gives a rheomorphic rather than causal description of the phenomenological laws: transport coefficients at an event x are taken to depend, not on the entire causal past of x, but only on the past history of a "comoving local fluid element". It is a linear theory restricted to small deviations from equilibrium, which can be derived from the linearized Boltzmann equation by projector-operator techniques (and probably inherits its causality properties). Instead of writing $(\delta n_\mu(x), \delta T_{\mu\nu}(x)) = \sigma(U, T)(-\partial_\mu \alpha(x), -\nabla_{(\mu}\beta_{\nu)}(x))$,

this local phenomenological law is generalized to $(\delta n_\mu(\vec{x}, x^o), \delta T_{\mu\nu}(\vec{x}, x^o)) = \int_{-\infty}^{\infty} dx^{o'} \sigma(x^o - x^{o'})(-\partial_\mu \alpha(\vec{x} x^{o'}), -\nabla_{(\mu}\beta_{\nu)}(\vec{x} x^{o'}))$.

C) Local non-linear extended thermodynamics (ET).

It is more relevant for relativistic astrophysics, where correlation and memory effects are not of primary interest and, instead, one needs a tractable and consistent transport theory coextensive at the macroscopic level with Boltzmann's equation. It is assumed that the second order term $Q^\mu(\delta n^\nu, \delta T^{\nu\rho}, ..)$ does not depend on auxiliary variables vanishing at equilibrium: this ansatz is the phenomenological equivalent of Grad's 14-moment approximation in kinetic theory. These theories are called "second-order theories" and many of them are analyzed in Ref. [46]; when the dissipative fluxes are subject to a conservation equation, these theories are called of causal "divergence type" like the ones of Refs. [47–49]. Another type of theory (extended irreversible thermodynamics; in general these theories are not of divergence type) was developed in Refs. [50–53]: in it there are transport equations for the dissipative fluxes rather than conservation laws.

For small deviations one retains only the quadratic terms in the Taylor expansion of Q^μ (leading to "linear" phenomenological laws): this implies 5 new undetermined coefficients

$$Q^\mu = \frac{1}{2}U^\mu[\beta_o \pi^2 + \beta_1 q^\mu q_\mu + \beta_2 \pi^{\mu\nu}\pi_{\mu\nu}] - \alpha_o \pi q^\mu - \alpha_1 \pi^{\mu\nu}q_\nu, \qquad (B34)$$

with $\beta_i > 0$ from $\epsilon U_\mu Q^\mu > 0$ (the β_i's are 'relaxation times'). A first-order change of rest frame produces a second-order change in Q^μ [going from the Landau-Lifshitz frame to the Eckart one, one gets $\alpha_{(eck)i} - \alpha_{(ll)i} = \beta_{(ll)1} - \beta_{(eck)1} = [(\rho + p)T]^{-1}$, $\beta_{(ll)0} = \beta_{(eck)o}$, $\beta_{(ll)2} = \beta_{(eck)2}$, and the phenomenological laws are now invariant to first order].

In the "Eckart frame" the phenomenological laws take the form

$$q^\mu_{(eck)} = -\kappa T(^4g^{\mu\nu} - \epsilon U^\mu_{(eck)}U^\nu_{(eck)})[T^{-1}\partial_\nu T + a_{(eck)\nu} + \beta_{(eck)1}\partial_\tau q_{(eck)\nu} -$$
$$- \alpha_{(eck)o}\partial_\nu \pi_{(eck)} - \alpha_{(eck)1}\nabla_\rho \pi^\rho_{(eck)\nu}],$$

$$\pi_{(eck)\mu\nu} = -2\zeta_S[< \nabla_\nu U_{(eck)\mu} > + \beta_{(eck)2}\partial_\tau \pi_{(eck)\mu\nu} - \alpha_{(eck)1} < \nabla_\nu q_{(eck)\mu} >],$$

$$\pi_{(eck)} = -\frac{1}{3}\zeta_V[\nabla_\mu U^\mu_{(eck)} + \beta_{(eck)o}\partial_\tau \pi_{(eck)} - \alpha_{(eck)o}\nabla_\mu q^\mu_{(eck)}], \qquad (B35)$$

which reduce to the equation of the standard Eckart theory if the 5 relaxation (β_i) and coupling (α_i) coefficients are put equal to zero. See for instance Ref. [54] for a complete treatment and also Ref. [55]. For appropriate values of these coefficients these equations are hyperbolic and, therefore, causal and stable. The transport equations can be understood [44] as evolution equations for the dissipative variables as they describe how these fluxes evolve from an initial arbitrary state to a final steady one [the time parameter τ is usually interpreted as the relaxation time of the dissipative processes]. In the case of a gas the new coefficients can be found explicitly [52] [see also Ref. [56] for a recent approach to relativistic

interacting gases starting from the Boltzmann equation], and they are purely thermody-namical functions. Wave front speeds are finite and comparable with the speed of sound. A problem with these theories is that they do not admit a regular shock structure (like the Navier-Stokes equations) once the speed of the shock front exceeds the highest characteristic velocity (a "subshock" will form within a shock layer for speeds exceeding the wave-front velocities of thermo-viscous effects). The situation slowly ameliorates if more moments are taken into account [54].

In the approach reviewed in Ref. [54] the extra indeterminacy associated to the new 5 coefficients is eliminated (at the price of high non-linearity) by annexing to the usual con-servation and entropy laws a new phenomenological assumption (in this way one obtains a causal divergence type theory):

$$\nabla_\rho A^{\mu\nu\rho} = I^{\mu\nu}, \tag{B36}$$

in which $A^{\rho\mu\nu}$ and $I^{\mu\nu}$ are symmetric tensors with the following traces

$$A^{\mu\nu}{}_\nu = -n^\mu, \qquad I^\mu{}_\mu = 0. \tag{B37}$$

These conditions are modelled on kinetic theory, in which $A^{\rho\mu\nu}$ represents the third mo-ment of the distribution function in momentum space, and $I^{\mu\nu}$ the second moment of the collision term in Boltzmann's equation. The previous equations are central in the determina-tion of the distribution function in Grad's 14-moment approximation. The phenomenologi-cal theory is completed by the postulate that the state variables S^μ, $A^{\rho\mu\nu}$, $I^{\mu\nu}$ are invariant functions of $T^{\mu\nu}$, n^μ only. The theory is an almost exact phenomenological counterpart of the Grad approximation. See Ref. [37] for the beginning (only non viscous heta conduct-ing materials are treated) of a derivation of extended thermodynamics from a variational principle.

Everything may be rephrased in terms of the Lagrangian coordinates of the fluid used in this paper. What is lacking in the non-dissipative case of heat conduction is the functional form of the off-equilibrium equation of state reducing to $\rho = \rho(n, s)$ at thermal equilibrium. In the dissipative case the system is open and $T^{\mu\nu}$, P^μ, n^μ are not conserved.

See Ref. [57] for attempts to define a classical theory of dissipation in the Hamiltonian framework and Ref. [58] about Hamiltonian molecular dynamics for the addition of an extra degree of freedom to an N-body system to transform it into an open system (with the choice of a suitable potential for the extra variable the equilibrium distribution function of the N-body subsystem is exactly the canonical ensemble).

However, the most constructive procedure is to get (starting from an action principle) the Hamiltonian form of the energy-momentum of a closed system, like it has been done in Ref. [17] for a system of N charged scalar particles, in which the mutual action-at-a-distance interaction is the complete Darwin potential extracted from the Lienard-Wiechert solution in the radiation gauge (the interactions are momentum- and, therefore, velocity-dependent). In this case one can define an open (in general dissipative) subsystem by considering a cluster of $n < N$ particles and assigning to it a non-conserved energy-momentum tensor built with all the terms of the original energy-momentum tensor which depend on the canonical variables of the n particles (the other $N - n$ particles are considered as external fields).

APPENDIX C: NOTATIONS ON SPACELIKE HYPERSURFACES.

Let us first review some preliminary results from Refs. [10] needed in the description of physical systems on spacelike hypersurfaces.

Let $\{\Sigma_\tau\}$ be a one-parameter family of spacelike hypersurfaces foliating Minkowski space-time M^4 with 4-metric $\eta_{\mu\nu} = \epsilon(+ - - -)$, $\epsilon = \pm$ [$\epsilon = +1$ is the particle physics convention; $\epsilon = -1$ the general relativity one] and giving a 3+1 decomposition of it. At fixed τ, let $z^\mu(\tau, \vec{\sigma})$ be the coordinates of the points on Σ_τ in M^4, $\{\vec{\sigma}\}$ a system of coordinates on Σ_τ. If $\sigma^{\check{A}} = (\sigma^\tau = \tau; \vec{\sigma} = \{\sigma^{\check{r}}\})$ [the notation $\check{A} = (\tau, \check{r})$ with $\check{r} = 1, 2, 3$ will be used; note that $\check{A} = \tau$ and $\check{A} = \check{r} = 1, 2, 3$ are Lorentz-scalar indices] and $\partial_{\check{A}} = \partial/\partial\sigma^{\check{A}}$, one can define the vierbeins

$$z^\mu_{\check{A}}(\tau, \vec{\sigma}) = \partial_{\check{A}} z^\mu(\tau, \vec{\sigma}), \qquad \partial_{\check{B}} z^\mu_{\check{A}} - \partial_{\check{A}} z^\mu_{\check{B}} = 0, \tag{C1}$$

so that the metric on Σ_τ is

$$g_{\check{A}\check{B}}(\tau, \vec{\sigma}) = z^\mu_{\check{A}}(\tau, \vec{\sigma})\eta_{\mu\nu} z^\nu_{\check{B}}(\tau, \vec{\sigma}), \qquad \epsilon g_{\tau\tau}(\tau, \vec{\sigma}) > 0,$$

$$g(\tau, \vec{\sigma}) = -det \, || \, g_{\check{A}\check{B}}(\tau, \vec{\sigma}) \, || = (det \, || \, z^\mu_{\check{A}}(\tau, \vec{\sigma}) \, ||)^2,$$

$$\gamma(\tau, \vec{\sigma}) = -det \, || \, g_{\check{r}\check{s}}(\tau, \vec{\sigma}) \, || = det \, ||^3 g_{\check{r}\check{s}}(\tau, \vec{\sigma})||, \tag{C2}$$

where $g_{\check{r}\check{s}} = -\epsilon\, ^3 g_{\check{r}\check{s}}$ with $^3 g_{\check{r}\check{s}}$ having positive signature $(+ + +)$.

If $\gamma^{\check{r}\check{s}}(\tau, \vec{\sigma}) = -\epsilon\, ^3 g^{\check{r}\check{s}}$ is the inverse of the 3-metric $g_{\check{r}\check{s}}(\tau, \vec{\sigma})$ [$\gamma^{\check{r}\check{u}}(\tau, \vec{\sigma})g_{\check{u}\check{s}}(\tau, \vec{\sigma}) = \delta^{\check{r}}_{\check{s}}$], the inverse $g^{\check{A}\check{B}}(\tau, \vec{\sigma})$ of $g_{\check{A}\check{B}}(\tau, \vec{\sigma})$ [$g^{\check{A}\check{C}}(\tau, \vec{\sigma})g_{\check{c}\check{b}}(\tau, \vec{\sigma}) = \delta^{\check{A}}_{\check{B}}$] is given by

$$g^{\tau\tau}(\tau, \vec{\sigma}) = \frac{\gamma(\tau, \vec{\sigma})}{g(\tau, \vec{\sigma})},$$

$$g^{\tau\check{r}}(\tau, \vec{\sigma}) = -[\frac{\gamma}{g} g_{\tau\check{u}} \gamma^{\check{u}\check{r}}](\tau, \vec{\sigma}) = \epsilon[\frac{\gamma}{g} g_{\tau\check{u}}\, ^3 g^{\check{u}\check{r}}](\tau, \vec{\sigma}),$$

$$g^{\check{r}\check{s}}(\tau, \vec{\sigma}) = \gamma^{\check{r}\check{s}}(\tau, \vec{\sigma}) + [\frac{\gamma}{g} g_{\tau\check{u}} g_{\tau\check{v}} \gamma^{\check{u}\check{r}} \gamma^{\check{v}\check{s}}](\tau, \vec{\sigma}) =$$

$$= -\epsilon\, ^3 g^{\check{r}\check{s}}(\tau, \vec{\sigma}) + [\frac{\gamma}{g} g_{\tau\check{u}} g_{\tau\check{v}}\, ^3 g^{\check{u}\check{r}}\, ^3 g^{\check{v}\check{s}}](\tau, \vec{\sigma}), \tag{C3}$$

so that $1 = g^{\tau\check{C}}(\tau, \vec{\sigma})g_{\check{C}\tau}(\tau, \vec{\sigma})$ is equivalent to

$$\frac{g(\tau, \vec{\sigma})}{\gamma(\tau, \vec{\sigma})} = g_{\tau\tau}(\tau, \vec{\sigma}) - \gamma^{\check{r}\check{s}}(\tau, \vec{\sigma})g_{\tau\check{r}}(\tau, \vec{\sigma})g_{\tau\check{s}}(\tau, \vec{\sigma}). \tag{C4}$$

We have

$$z^\mu_\tau(\tau, \vec{\sigma}) = (\sqrt{\frac{g}{\gamma}} l^\mu + g_{\tau\check{r}} \gamma^{\check{r}\check{s}} z^\mu_{\check{s}})(\tau, \vec{\sigma}), \tag{C5}$$

and

$$\eta^{\mu\nu} = z^\mu_{\check{A}}(\tau, \vec{\sigma})g^{\check{A}\check{B}}(\tau, \vec{\sigma})z^\nu_{\check{B}}(\tau, \vec{\sigma}) =$$

$$= (l^\mu l^\nu + z^\mu_{\check{r}} \gamma^{\check{r}\check{s}} z^\nu_{\check{s}})(\tau, \vec{\sigma}), \tag{C6}$$

where

$$l^\mu(\tau, \vec{\sigma}) = (\frac{1}{\sqrt{\gamma}} \epsilon^\mu{}_{\alpha\beta\gamma} z_{\check{1}}^\alpha z_{\check{2}}^\beta z_{\check{3}}^\gamma)(\tau, \vec{\sigma}),$$
$$l^2(\tau, \vec{\sigma}) = 1, \qquad l_\mu(\tau, \vec{\sigma}) z_{\check{r}}^\mu(\tau, \vec{\sigma}) = 0, \tag{C7}$$

is the unit (future pointing) normal to Σ_τ at $z^\mu(\tau, \vec{\sigma})$.

For the volume element in Minkowski spacetime we have

$$d^4z = z_\tau^\mu(\tau, \vec{\sigma}) d\tau d^3\Sigma_\mu = d\tau[z_\tau^\mu(\tau, \vec{\sigma}) l_\mu(\tau, \vec{\sigma})]\sqrt{\gamma(\tau, \vec{\sigma})} d^3\sigma =$$
$$= \sqrt{g(\tau, \vec{\sigma})} d\tau d^3\sigma. \tag{C8}$$

Let us remark that according to the geometrical approach of Ref. [12],one can use Eq.(C5) in the form

$$z_\tau^\mu(\tau, \vec{\sigma}) = N(\tau, \vec{\sigma}) l^\mu(\tau, \vec{\sigma}) + N^{\check{r}}(\tau, \vec{\sigma}) z_{\check{r}}^\mu(\tau, \vec{\sigma}),$$

where $N = \sqrt{g/\gamma} = \sqrt{g_{\tau\tau} - \gamma^{\check{r}\check{s}} g_{\tau\check{r}} g_{\tau\check{s}}} = \sqrt{g_{\tau\tau} + \epsilon^3 g^{\check{r}\check{s}} g_{\tau\check{r}} g_{\tau\check{s}}}$ and $N^{\check{r}} = g_{\tau\check{s}} \gamma^{\check{s}\check{r}} = -\epsilon g_{\tau\check{s}}{}^3 g^{\check{s}\check{r}}$

are the standard lapse and shift functions $N_{[z](flat)}$, $N^{\check{r}}_{[z](flat)}$ of the Introduction, so that

$$g_{\tau\tau} = \epsilon N^2 + g_{\check{r}\check{s}} N^{\check{r}} N^{\check{s}} = \epsilon[N^2 - {}^3 g_{\check{r}\check{s}} N^{\check{r}} N^{\check{s}}],$$
$$g_{\tau\check{r}} = g_{\check{r}\check{s}} N^{\check{s}} = -\epsilon^3 g_{\check{r}\check{s}} N^{\check{s}},$$
$$g^{\tau\tau} = \epsilon N^{-2},$$
$$g^{\tau\check{r}} = -\epsilon N^{\check{r}}/N^2,$$
$$g^{\check{r}\check{s}} = \gamma^{\check{r}\check{s}} + \epsilon \frac{N^{\check{r}} N^{\check{s}}}{N^2} = -\epsilon[{}^3 g^{\check{r}\check{s}} - \frac{N^{\check{r}} N^{\check{s}}}{N^2}],$$

$$\frac{\partial}{\partial z_\tau^\mu} = l_\mu \frac{\partial}{\partial N} + z_{\check{s}\mu} \gamma^{\check{s}\check{r}} \frac{\partial}{\partial N^{\check{r}}} = l_\mu \frac{\partial}{\partial N} - \epsilon z_{\check{s}\mu}{}^3 g^{\check{s}\check{r}} \frac{\partial}{\partial N^{\check{r}}},$$
$$d^4z = N\sqrt{\gamma} d\tau d^3\sigma.$$

The rest frame form of a timelike fourvector p^μ is $\overset{\circ}{p}{}^\mu = \eta\sqrt{\epsilon p^2}(1; \vec{0}) = \eta^{\mu o}\eta\sqrt{\epsilon p^2}$, $\overset{\circ}{p}{}^2 = p^2$, where $\eta = sign\, p^o$. The standard Wigner boost transforming $\overset{\circ}{p}{}^\mu$ into p^μ is

$$L^\mu{}_\nu(p, \overset{\circ}{p}) = \epsilon_\nu^\mu(u(p)) =$$
$$= \eta_\nu^\mu + 2\frac{p^\mu \overset{\circ}{p}_\nu}{\epsilon p^2} - \frac{(p^\mu + \overset{\circ}{p}{}^\mu)(p_\nu + \overset{\circ}{p}_\nu)}{p \cdot \overset{\circ}{p} + \epsilon p^2} =$$
$$= \eta_\nu^\mu + 2u^\mu(p) u_\nu(\overset{\circ}{p}) - \frac{(u^\mu(p) + u^\mu(\overset{\circ}{p}))(u_\nu(p) + u_\nu(\overset{\circ}{p}))}{1 + u^o(p)},$$

$$\nu = 0 \quad \epsilon_o^\mu(u(p)) = u^\mu(p) = p^\mu/\eta\sqrt{\epsilon p^2},$$
$$\nu = r \quad \epsilon_r^\mu(u(p)) = (-u_r(p); \delta_r^i - \frac{u^i(p) u_r(p)}{1 + u^o(p)}). \tag{C9}$$

The inverse of $L^\mu{}_\nu(p, \overset{\circ}{p})$ is $L^\mu{}_\nu(\overset{\circ}{p}, p)$, the standard boost to the rest frame, defined by

$$L^{\mu}{}_{\nu}(\overset{\circ}{p},p) = L_{\nu}{}^{\mu}(p,\overset{\circ}{p}) = L^{\mu}{}_{\nu}(p,\overset{\circ}{p})|_{\vec{p}\to-\vec{p}}. \tag{C10}$$

Therefore, we can define the following vierbeins [the $\epsilon_r^{\mu}(u(p))$'s are also called polarization vectors; the indices r, s will be used for A=1,2,3 and $\bar{o}$ for $A = o$]

$$\epsilon_A^{\mu}(u(p)) = L^{\mu}{}_A(p,\overset{\circ}{p}),$$
$$\epsilon_{\mu}^A(u(p)) = L^A{}_{\mu}(\overset{\circ}{p},p) = \eta^{AB}\eta_{\mu\nu}\epsilon_B^{\nu}(u(p)),$$

$$\epsilon_{\mu}^{\bar{o}}(u(p)) = \eta_{\mu\nu}\epsilon_o^{\nu}(u(p)) = u_{\mu}(p),$$
$$\epsilon_{\mu}^r(u(p)) = -\delta^{rs}\eta_{\mu\nu}\epsilon_r^{\nu}(u(p)) = (\delta^{rs}u_s(p); \delta_j^r - \delta^{rs}\delta_{jh}\frac{u^h(p)u_s(p)}{1+u^o(p)}),$$
$$\epsilon_o^A(u(p)) = u_A(p), \tag{C11}$$

which satisfy

$$\epsilon_{\mu}^A(u(p))\epsilon_A^{\nu}(u(p)) = \eta_{\nu}^{\mu},$$
$$\epsilon_{\mu}^A(u(p))\epsilon_B^{\mu}(u(p)) = \eta_B^A,$$
$$\eta^{\mu\nu} = \epsilon_A^{\mu}(u(p))\eta^{AB}\epsilon_B^{\nu}(u(p)) = u^{\mu}(p)u^{\nu}(p) - \sum_{r=1}^{3}\epsilon_r^{\mu}(u(p))\epsilon_r^{\nu}(u(p)),$$
$$\eta_{AB} = \epsilon_A^{\mu}(u(p))\eta_{\mu\nu}\epsilon_B^{\nu}(u(p)),$$
$$p_{\alpha}\frac{\partial}{\partial p_{\alpha}}\epsilon_A^{\mu}(u(p)) = p_{\alpha}\frac{\partial}{\partial p_{\alpha}}\epsilon_{\mu}^A(u(p)) = 0. \tag{C12}$$

The Wigner rotation corresponding to the Lorentz transformation Λ is

$$R^{\mu}{}_{\nu}(\Lambda,p) = [L(\overset{\circ}{p},p)\Lambda^{-1}L(\Lambda p,\overset{\circ}{p})]^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 \\ 0 & R^i{}_j(\Lambda,p) \end{pmatrix},$$

$$R^i{}_j(\Lambda,p) = (\Lambda^{-1})^i{}_j - \frac{(\Lambda^{-1})^i{}_o p_{\beta}(\Lambda^{-1})^{\beta}{}_j}{p_{\rho}(\Lambda^{-1})^{\rho}{}_o + \eta\sqrt{\epsilon p^2}} -$$
$$- \frac{p^i}{p^o + \eta\sqrt{\epsilon p^2}}[(\Lambda^{-1})^o{}_j - \frac{((\Lambda^{-1})^o{}_o - 1)p_{\beta}(\Lambda^{-1})^{\beta}{}_j}{p_{\rho}(\Lambda^{-1})^{\rho}{}_o + \eta\sqrt{\epsilon p^2}}]. \tag{C13}$$

The polarization vectors transform under the Poincaré transformations (a, Λ) in the following way

$$\epsilon_r^{\mu}(u(\Lambda p)) = (R^{-1})_r{}^s \Lambda^{\mu}{}_{\nu} \epsilon_s^{\nu}(u(p)). \tag{C14}$$

APPENDIX D: MORE ON DIXON'S MULTIPOLES.

Let us add other forms of the Dixon multipoles.

In the case of the fluid configurations treated in Section II and IV, the Hamilton equations generated by the Dirac Hamiltonian (2.49) in the gauge $\vec{q}_{sys} \approx 0$ $[\vec{\lambda}(\tau) = 0]$ imply [in Ref. [20] this is a consequence of $\partial_\mu T^{\mu\nu} \overset{\circ}{=} 0$]

$$\frac{dp_T^\mu(T_s)}{dT_s} \overset{\circ}{=} 0, \quad for\, n = 0,$$

$$\frac{dp_T^{\mu_1\cdots\mu_n\mu}(T_s)}{dT_s} \overset{\circ}{=} -nu^{(\mu_1}(p_s)p_T^{\mu_2\cdots\mu_n)\mu}(T_s) + nt_T^{(\mu_1\cdots\mu_n)\mu}(T_s), \quad n \geq 1. \tag{D1}$$

Let us define for $n \geq 1$

$$b_T^{\mu_1\cdots\mu_n\mu}(T_s) = p_T^{(\mu_1\cdots\mu_n\mu)}(T_s) =$$
$$= \epsilon_{r_1}^{(\mu_1}(u(p_s))....\epsilon_{r_n}^{\mu_n}(u(p_s))\epsilon_A^{\mu)}(u(p_s))I_T^{r_1..r_nA\tau}(T_s),$$

$$\epsilon_{\mu_1}^{r_1}(u(p_s))....\epsilon_{\mu_n}^{r_n}(u(p_s))b_T^{\mu_1\cdots\mu_n\mu}(T_s) = \frac{1}{n+1}u^\mu(p_s)I_T^{r_1\cdots r_n\tau\tau}(T_s) + \epsilon_r^\mu(u(p_s))I_T^{(r_1\cdots r_nr)\tau}(T_s),$$

$$c_T^{\mu_1\cdots\mu_n\mu}(T_s) = c_T^{(\mu_1\cdots\mu_n)\mu}(T_s) = p_T^{\mu_1\cdots\mu_n\mu}(T_s) - p_T^{(\mu_1\cdots\mu_n\mu)}(T_s) =$$
$$= [\epsilon_{r_1}^{\mu_1}(u(p_s))...\epsilon_{r_n}^{\mu_n}\epsilon_A^\mu(u(p_s)) -$$
$$- \epsilon_{r_1}^{(\mu_1}(u(p_s))...\epsilon_{r_n}^{\mu_n}(u(p_s))\epsilon_A^{\mu)}(u(p_s))]I_T^{r_1..r_nA\tau}(T_s),$$

$$c_T^{(\mu_1\cdots\mu_n\mu)}(T_s) = 0,$$

$$\epsilon_{\mu_1}^{r_1}(u(p_s))....\epsilon_{\mu_n}^{r_n}(u(p_s))c_T^{\mu_1\cdots\mu_n\mu}(T_s) = \frac{n}{n+1}u^\mu(p_s)I_T^{r_1\cdots r_n\tau\tau}(T_s) +$$
$$+ \epsilon_r^\mu(u(p_s))[I_T^{r_1\cdots r_nr\tau}(T_s) - I_T^{(r_1\cdots r_nr)\tau}(T_s)], \tag{D2}$$

and then for $n \geq 2$

$$d_T^{\mu_1\cdots\mu_n\mu\nu}(T_s) = d_T^{(\mu_1\cdots\mu_n)(\mu\nu)}(T_s) = t_T^{\mu_1\cdots\mu_n\mu\nu}(T_s) -$$
$$- \frac{n+1}{n}[t_T^{(\mu_1\cdots\mu_n\mu)\nu}(T_s) + t_T^{(\mu_1\cdots\mu_n\nu)\mu}(T_s)] +$$
$$+ \frac{n+2}{n}t_T^{(\mu_1\cdots\mu_n\mu\nu)}(T_s) =$$
$$= \Big[\epsilon_{r_1}^{\mu_1}...\epsilon_{r_n}^{\mu_n}\epsilon_A^\mu\epsilon_B^\nu - \frac{n+1}{n}\Big(\epsilon_{r_1}^{(\mu_1}...\epsilon_{r_n}^{\mu_n}\epsilon_A^{\mu)}\epsilon_B^\nu +$$
$$+ \epsilon_{r_1}^{(\mu_1}...\epsilon_{r_n}^{\mu_n}\epsilon_B^{\nu)}\epsilon_A^\mu\Big) + \frac{n+2}{n}\epsilon_{r_1}^{(\mu_1}..\epsilon_{r_n}^{\mu_n}\epsilon_A^\mu\epsilon_B^{\nu)}\Big](u(p_s))$$
$$I_T^{r_1..r_nAB}(T_s),$$

$$d_T^{(\mu_1\cdots\mu_n\mu)\nu}(T_s) = 0,$$

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$$\epsilon^{r_1}_{\mu_1}(u(p_s))....\epsilon^{r_n}_{\mu_n}(u(p_s))d_T^{\mu_1...\mu_n\mu\nu}(T_s) = \frac{n-1}{n+1}u^\mu(p_s)u^\nu(p_s)I_T^{r_1...r_n\tau\tau}(T_s) +$$

$$+ \frac{1}{n}[u^\mu(p_s)\epsilon^\nu_r(u(p_s)) + u^\nu(p_s)\epsilon^\mu_r(u(p_s))]$$

$$[(n-1)I_T^{r_1...r_nr\tau}(T_s) + I_T^{(r_1...r_nr)\tau}(T_s)] +$$

$$+ \epsilon^\mu_{s_1}(u(p_s))\epsilon^\nu_{s_2}(u(p_s))[I_T^{r_1...r_ns_1s_2}(T_s) -$$

$$- \frac{n+1}{n}(I_T^{(r_1...r_ns_1)s_2}(T_s) + I_T^{(r_1...r_ns_2)s_1}(T_s)) +$$

$$+ I_T^{(r_1...r_ns_1s_2)}(T_s)]. \tag{D3}$$

Then Eqs.(D1) may be rewritten in the form

$$1) \quad n = 1$$

$$t_T^{\mu\nu}(T_s) = t_T^{(\mu\nu)}(T_s) \overset{\circ}{=} p_T^\mu(T_s)u^\nu(p_s) + \frac{1}{2}\frac{d}{dT_s}(S_T^{\mu\nu}(T_s)[\alpha] + 2b_T^{\mu\nu}(T_s)),$$

$$\Downarrow$$

$$t_T^{\mu\nu}(T_s) \overset{\circ}{=} p_T^{(\mu}(T_s)u^{\nu)}(p_s) + \frac{d}{dT_s}b_T^{\mu\nu}(T_s) = P^\tau u^\mu(p_s)u^\nu(p_s) + P^r u^{(\mu}(p_s)\epsilon^{\nu)}_r(u(p_s)) +$$

$$+ \epsilon^{(\mu}_r(u(p_s))u^{\nu)}(p_s)I_T^{r\tau\tau}(T_s) +$$

$$+ \epsilon^{(\mu}_r(u(p_s))\epsilon^{\nu)}_s(u(p_s))I_T^{rs\tau}(T_s),$$

$$\frac{d}{dT_s}S_T^{\mu\nu}(T_s)[\alpha] \overset{\circ}{=} 2p_T^{[\mu}(T_s)u^{\nu]}(p_s) = 2P^r_\phi\epsilon^{[\mu}_r(u(p_s))u^{\nu]}(p_s) \approx 0,$$

$$2) \quad n = 2 \quad [identity \ t_T^{\rho\mu\nu} = t_T^{(\rho\mu)\nu} + t_T^{(\rho\nu)\mu} + t_T^{(\mu\nu)\rho}]$$

$$2t_T^{(\rho\mu)\nu}(T_s) \overset{\circ}{=} 2u^{(\rho}(p_s)b_T^{\mu)\nu}(T_s) + u^{(\rho}(p_s)S_T^{\mu)\nu}(T_s)[\alpha] + \frac{d}{dT_s}(b_T^{\rho\mu\nu}(T_s) + c_T^{\rho\mu\nu}(T_s)),$$

$$\Downarrow$$

$$t_T^{\rho\mu\nu}(T_s) \overset{\circ}{=} u^\rho(p_s)b_T^{\mu\nu}(T_s) + S_T^{\rho(\mu}(T_s)[\alpha]u^{\nu)}(p_s) + \frac{d}{dT_s}(\frac{1}{2}b_T^{\rho\mu\nu}(T_s) - c_T^{\rho\mu\nu}(T_s)),$$

$$3) \quad n \geq 3$$

$$t_T^{\mu_1...\mu_n\mu\nu}(T_s) \overset{\circ}{=} d_T^{\mu_1...\mu_n\mu\nu}(T_s) + u^{(\mu_1}(p_s)b_T^{\mu_2...\mu_n)\mu\nu}(T_s) + 2u^{(\mu_1}(p_s)c_T^{\mu_2...\mu_n)(\mu\nu)}(T_s) +$$

$$= \frac{2}{n}c_T^{\mu_1...\mu_n(\mu}(T_s)u^{\nu)}(p_s) + \frac{d}{dT_s}[\frac{1}{n+1}b_T^{\mu_1...\mu_n\mu\nu}(T_s) + \frac{2}{n}c_T^{\mu_1...\mu_n(\mu\nu)}(T_s)], \tag{D4}$$

This allows [20] to rewrite $< T^{\mu\nu}, f >$ in the following form

$$< T^{\mu\nu}, f > = \int dT_s \int \frac{d^4k}{(2\pi)^4}\tilde{f}(k)e^{-ik\cdot x_s(T_s)}\Big[u^{(\mu}(p_s)p_T^{\nu)}(T_s) - ik_\rho S_T^{\rho(\mu}(T_s)[\alpha]u^{\nu)}(p_s) +$$

$$+ \sum_{n=2}^{\infty}\frac{(-i)^n}{n!}k_{\rho_1}...k_{\rho_n}\mathcal{I}_T^{\rho_1...\rho_n\mu\nu}(T_s)\Big], \tag{D5}$$

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with

$$
\mathcal{I}_T^{\mu_1\dots\mu_n\mu\nu}(T_s) = \mathcal{I}_T^{(\mu_1\dots\mu_n)(\mu\nu)}(T_s) = d_T^{\mu_1\dots\mu_n\mu\nu}(T_s) -
$$

$$
- \frac{2}{n-1}u^{(\mu_1}(p_s)c_T^{\mu_2\dots\mu_n)(\mu\nu)}(T_s) + \frac{2}{n}c_T^{\mu_1\dots\mu_n(\mu}(T_s)u^{\nu)}(p_s) =
$$

$$
= \Big[\epsilon_{r_1}^{\mu_1}\dots\epsilon_{r_n}^{\mu_n}\epsilon_A^{\mu}\epsilon_B^{\nu} - \frac{n+1}{n}\Big(\epsilon_{r_1}^{(\mu_1}\dots\epsilon_{r_n}^{\mu_n}\epsilon_A^{\mu)}\epsilon_B^{\nu} +
$$

$$
+ \epsilon_{r_1}^{(\mu_1}\dots\epsilon_{r_n}^{\mu_n}\epsilon_B^{\nu)}\epsilon_A^{\mu}\Big) + \frac{n+2}{n}\epsilon_{r_1}^{(\mu_1}\dots\epsilon_{r_n}^{\mu_n}\epsilon_A^{\mu}\epsilon_B^{\nu)}\Big](u(p_s)0I_T^{r_1..r_nAB}(T_s) -
$$

$$
- \Big[\frac{2}{n-1}u^{(\mu_1}(p_s)\Big(\epsilon_{r_1}^{\mu_2}\dots\epsilon_{r_{n-1}}^{\mu_n)}\epsilon_{r_n}^{(\mu}\epsilon_A^{\nu)} - \epsilon_{r_1}^{(\mu_2}\dots\epsilon_{r_{n-1}}^{\mu_n)}\epsilon_{r_n}^{(\mu}\epsilon_A^{\nu))}\Big) -
$$

$$
- \frac{2}{n}\Big(\epsilon_{r_1}^{\mu_1}\dots\epsilon_{r_n}^{\mu_n}\epsilon_A^{(\mu} - \epsilon_{r_1}^{(\mu_1}\dots\epsilon_{r_n}^{\mu_n}\epsilon_A^{(\mu)}u^{\nu)}(p_s)\Big](u(p_s)0I_T^{r_1..r_nA\tau}(T_s),
$$

$$
\mathcal{I}_T^{(\mu_1\dots\mu_n\mu)\nu}(T_s) = 0,
$$

$$
\epsilon_{\mu_1}^{r_1}(u(p_s))\dots\epsilon_{\mu_n}^{r_n}(u(p_s))\mathcal{I}_T^{\mu_1\dots\mu_n\mu\nu}(T_s) = \frac{n+3}{n+1}u^{\mu}(p_s)u^{\nu}(p_s)I_T^{r_1\dots r_n\tau\tau}(T_s) +
$$

$$
+ \frac{1}{n}[u^{\mu}(p_s)\epsilon_r^{\nu}(u(p_s)) + u^{\nu}(p_s)\epsilon_r^{\mu}(u(p_s))]I_T^{r_1\dots r_nr\tau}(T_s) +
$$

$$
+ \epsilon_{s_1}^{\mu}(u(p_s))\epsilon_{s_2}^{\nu}(u(p_s))[I_T^{r_1\dots r_ns_1s_2}(T_s) -
$$

$$
- \frac{n+1}{n}(I_T^{(r_1\dots r_ns_1)s_2}(T_s) + I_T^{(r_1\dots r_ns_2)s_1}(T_s)) +
$$

$$
+ I_T^{(r_1\dots r_ns_1s_2)}(T_s)]. \tag{D6}
$$

Finally, a set of multipoles equivalent to the $\mathcal{I}_T^{\mu_1\dots\mu_n\mu\nu}$ is

$$
n \geq 0
$$

$$
J_T^{\mu_1\dots\mu_n\mu\nu\rho\sigma}(T_s) = J_T^{(\mu_1\dots\mu_n)[\mu\nu][\rho\sigma]}(T_s) = \mathcal{I}_T^{\mu_1\dots\mu_n[\mu[\rho\nu]\sigma]}(T_s) =
$$

$$
= t_T^{\mu_1\dots\mu_n[\mu[\rho\nu]\sigma]}(T_s) - \frac{1}{n+1}\Big[u^{[\mu}(p_s)p_T^{\nu]\mu_1\dots\mu_n[\rho\sigma]}(T_s) +
$$

$$
+ u^{[\rho}(p_s)p_T^{\sigma]\mu_1\dots\mu_n[\mu\nu]}(T_s)\Big] =
$$

$$
= \Big[\epsilon_{r_1}^{\mu_1}..\epsilon_{r_n}^{\mu_n}\epsilon_r^{[\mu}\epsilon_s^{[\rho}\epsilon_A^{\nu]}\epsilon_B^{\sigma]}\Big](u(p_s))I_T^{r_1..r_nAB}(T_s) -
$$

$$
- \frac{1}{n+1}\Big[u^{[\mu}(p_s)\epsilon_r^{\nu]}(u(p_s))\epsilon_s^{[\rho}(u(p_s))\epsilon_A^{\sigma]}(u(p_s)) +
$$

$$
+ u^{[\rho}(p_s)\epsilon_r^{\sigma]}(u(p_s))\epsilon_s^{[\mu}(u(p_s))\epsilon_A^{\nu]}(u(p_s))\Big]
$$

$$
\epsilon_{r_1}^{\mu_1}(u(p_s))\dots\epsilon_{r_n}^{\mu_n}(u(p_s))I_T^{rr_1..r_nsA\tau}(T_s),
$$

$$
[(n+4)(3n+5)\ linearly\ independent\ components],
$$

$$
n \geq 1
$$

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$$u_{\mu_1}(p_s) \quad J_T^{\mu_1 \ldots \mu_n \mu \nu \rho \sigma}(T_s) = J_T^{\mu_1 \ldots \mu_{n-1}(\mu_n \mu \nu)\rho\sigma}(T_s) = 0,$$

$$n \geq 2$$

$$\mathcal{I}_T^{\mu_1 \ldots \mu_n \mu \nu}(T_s) = \frac{4(n-1)}{n+1} J_T^{(\mu_1 \ldots \mu_{n-1}|\mu|\mu_n)\nu}(T_s),$$

$$\epsilon_{\mu_1}^{r_1}(u(p_s)) \ldots \epsilon_{\mu_n}^{r_n}(u(p_s)) J_T^{\mu_1 \ldots \mu_n \mu \nu \rho \sigma}(T_s) = \left[\epsilon_r^{[\mu} \epsilon_s^{[\rho} \epsilon_A^{\nu]} \epsilon_B^{\sigma]} \right] (u(p_s)) I_T^{r_1 \ldots r_n AB}(T_s) -$$

$$- \frac{1}{n+1} \left[u^{[\mu}(p_s) \epsilon_r^{\nu]}(u(p_s)) \epsilon_s^{[\rho}(u(p_s)) \epsilon_A^{\sigma]}(u(p_s)) + \right.$$

$$\left. + u^{[\rho}(p_s) \epsilon_r^{\sigma]}(u(p_s)) \epsilon_s^{[\mu}(u(p_s)) \epsilon_A^{\nu]}(u(p_s)) \right] I_T^{r r_1 \ldots r_n s A \tau}(T_s).$$

$$(D7)$$

The $J_T^{\mu_1 \ldots \mu_n \mu \nu \rho \sigma}$ are the Dixon "2^{n+2}-pole inertial moment tensors" of the extended system: they [or equivalently the $\mathcal{I}_T^{\mu_1 \ldots \mu_n \mu \nu}$'s] determine its energy-momentum tensor together with the monopole p_T^μ and the spin dipole $S_T^{\mu\nu}$. The equations $\partial_\mu T^{\mu\nu} \overset{\circ}{=} 0$ are satisfied due to the equations of motion (D4) for P_T^μ and $S_T^{\mu\nu}$ [the so called Papapetrou-Dixon-Souriau equations given in Eqs.(4.18)] without the need of the equations of motion for the $J_T^{\mu_1 \ldots \mu_n \mu \nu \rho \sigma}$. When all the multipoles $J_T^{\mu_1 \ldots \mu_n \mu \nu \rho \sigma}$ are zero [or negligible] one speaks of a pole-dipole field configuration of the perfect fluid.

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